

# THE KINEMATIC ANALYSIS AND SYNTHESIS OF CONIC CONSTRAINT PAIRS IN COPLANAR MOTION

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A DISSERTATION SUBMITTED TO THE GRADUATE SCHOOL OF  
THE UNIVERSITY OF FLORIDA  
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE  
DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA  
1978

for William Wesley Spaulding  
and Henry Lynn Spaulding

## ACKNOWLEDGEMENTS

The author expresses his appreciation to Dr. William F. Fox for the suggestion and supervision of this administration.

His thanks are extended to C.A. Williams and J.A. Bessinger for the financial support provided by their generous donations with the Southern Service Company and the Office of Civil Defense.

Gratitude is given to the following committee members for their guidance and supervision.

Dr. G. Foss

Dr. R. B. Galtner

Dr. J. H. Hargis

Dr. J. M. Vance

Dr. H. G. Selfridge.

Finally, the author extends his deepest appreciation to his father and mother, to his family and to his wife, Cheryl, for their financial support and encouragement.

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THESIS OF MEMORANDUM Presented to the Graduate  
Council in Partial Fulfillment of the Requirements for the  
Degree of Doctor of Philosophy

THE KINEMATIC ANALYSIS AND SYNTHESIS  
OF COUPLER CONSTRAINT BRIMS  
IN COUPLER MOTION

By

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August, 1972

Organized by Robert Dean  
Major Department: Mechanical Engineering

A comprehensive study of the analysis and synthesis of coupler constraint brims in coupler motion is presented in a generalized form. A state of the art is given for kinematic theory and for the generalized coupler theory. A seven position algorithm for the generalized coupler theory provides two equations relating the unknown generalized coupler points.

The analysis algorithm is shown to satisfy both the Denavit mechanism and the coupler mechanisms and does not require iterative techniques for solution. The conversion to real time is shown to be an elementary transformation.

Algorithms for prescribing fixed and moving pivots while satisfying five position points provide extensions to the well-known graphical procedures of the past. These

algorithms are presented for both Burmeister and Gossio Theory.

An extension of Burmeister Theory to the synthesis of six point-positions of the moving plane is presented with a theorem of uniqueness for six positions of the moving plane. The nine position point problem is reduced to a solution of five equations with five unknown variables.

Tabulated coefficients for the analytic and synthesis algorithms provide a systematic and simplified means of computation for all case studies.

## CHAPTER 2

### GENERAL BACKGROUND

In 1947 the author completed research which culminated the study of finite (positional) and infinitesimal (continuous) kinematics by presenting an algorithm for five multiply separated positions (any multiple combination of finite and/or infinitesimal positions) in explicit motion. The work is similar to that outlined by Reuleaux (1) for the finite case and requires the resolution of four linear and two non-linear algebraic equations based on circular constraints in the fixed plane. A generalized computer program capable of solving all seven cases was the culmination of this work. Since its completion there have been few research publications associated with generalized positional synthesis for mechanisms or frame assemblies. Consequently, as is being displayed at many kinematics symposia to have created a platform in the capabilities in explicit synthesis. Often when this happens the only thrust to follow is to survey the field of discipline with new conceptual tools to see if there might precipitate different concepts which will reflect upon and advance the particular area of study. Success is primarily dependent upon taking an interrogative viewpoint in the light of what is known and what has created such a

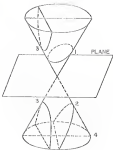
congruent. This technique generally allows one to gain new insight concerning his problem and most often establishes a solution.

The technique employed in synthesizing six point-positions and nine precision points, presented herein, is a result of taking an interrogative look at coplanar synthesis and applies a new concept to obtain the remaining algorithms. The algorithms presented for six point-positions and for nine precision points are capable of solving all case studies for multiply separated point-positions in coplanar motion.

### Conic Sections

It was René Descartes who in 1637 first applied algebra to geometry making the study of conic sections a part of elementary mathematics. As shown by Descartes, there are three general classes of curves obtained by the intersection of a plane and a cone as illustrated in Figure (1-1) and defined as follows:

- 1) An ellipse is formed when the plane section cuts only one nappe of a cone and is not parallel to an element of the cone. A circle is obtained as a special ellipse when the plane is perpendicular to the axis and results in a point at the vertex.
- 2) A parabola is obtained as a plane section becomes parallel to an element of the cone. A line is formed as a limiting case.
- 3) A hyperbola is formed in both nappes of the cone. Two intersecting lines are obtained as a limiting case.



### TYPES OF CONICS

- 1) ELLIPSE
- 2) PARABOLA
- 3) HYPERBOLA
- 4) CIRCLE (SPECIAL ELLIPSE)

FIGURE 10-1 Plane Intersecting a Cone in Space

### Analysis of the Coupler Curve Function for Single Revolute Joint

In recent years many kinematic researchers have reported considerable effort to define and analyze the output function (coupler curves) of the single four-bar mechanism. Since this study is concerned with single constrained joints in motion, an algorithm is presented to conceive and generalize the analysis of the single mechanism with the mode of synthesis.

As will be shown in the text, the analysis of the circular constrained system employs the same algorithm as the generalized curve constrained system. However, it becomes advantageous to discuss the two systems separately because of the difference in the complexity of the solutions.

The basic concern for the development of an analysis algorithm was to acquire a method of verifying the results provided by the synthesis program. The author found that his efforts were markedly reduced as a result of this particular investigation and formulation.

### Analysis of the Coupler Curve Function for Revolute Constrained

From a kinematic standpoint, analysis of coplanar linkages is certainly a well-known area of study. Works by Freudenstein [2] have provided explicit results for all types of planar (revolute) constrained systems. Therefore, it is not the intention of this paper to initiate new areas of research for analysis but simply to develop a

prescribed as  $\dot{\theta}_1$  with its corresponding with the prescribed mode of synthesis. However, it is discussed for analyzing positions as well as orders of contact (disrupt, recontact, etc.) for barometer constraints.

### Kinematic Synthesis for Barometer and Contact Constraints

Recent works by Sparto and Tzeng [3] have provided a mode of synthesizing a barometer pair while work by Hsu [4] have provided a graphical mode of synthesizing the ground link for barometer constraint pairs. It is the author's intention to provide a method which includes the above results and also provides the capability of synthesizing a coupler link or ground link while satisfying five prescribed multiply separated position points for barometer constraints.

A similar algorithm for synthesizing seven multiply separated position points is presented for the generalized mode pairs. It might be noted that for some constraint pairs, one can only synthesize the coupler link since contact (assuming circular constraints) have no physical center physically represented by a fixed pivot.

One might say that the dependence of dynamic synthesis lies solely within this area of research since one must have at least one constraint pair to synthesize the absolute dynamic characteristic. This particular area of research represents a division/discontinuity of kinematic synthesis and consequently differentiation to kinematic utilization.

### Diagramatic Synthesis for Separated Constraints

At present several different methods are employed for synthesizing 2,4 and 5 cognate positions of the moving plane [1,2,3,4,5,6]. Here, emphasis will be placed only on synthesis of 5 cognate positions since this problem represents the present capability of cognate synthesis. Early works by Reuleur [4] have placed the present capability for certain more studies at six precision points with somewhat limited application and questionable versatility as it often the consequence with graphical procedures.<sup>1</sup> An analytical approach by Freudenstein [2] has provided some implicit results with iterative techniques which often can be time consuming and fruitless.

It is therefore the intention of this paper to provide algorithms for synthesizing six mutually separated points-positions and six multiply separated precision points in cognate motion for Burmester constraints.

### Diagramatic Synthesis for Cognate Constraints

Considering cognate constraints when synthesizing cognate positions adds two positions to the capability represented by Burmester constraints. Present work by Freudenstein, Holtena and Kutzler [3] has provided a useful foundation for cognate constraints and has made the

<sup>1</sup>It must be noted that a position of the moving plane is prescribed by  $x, y$  (Cartesian) coordinates of the origin of the moving plane; and  $\gamma$  (angle of the moving plane) where a precision point is prescribed only by the coordinates  $x$  and  $y$ .

formulation were restricted to the solution of value problems.

This report will provide an abstraction for synthesizing some coplanar problems involving some constraint prices. The primary objective of this work will not be to develop an algorithm for synthesizing post-positions using some constraints, as in the case of Bernhart's constraints, due to the complexity of such an endeavor. The author will, however, provide answers to such research and therefore provide a foundation for future research.

The Application of  
Constraint Synthesis

The areas of application of coplanar linkages have recently become very broad and divergent. This is primarily a result of requirements imposed upon design by competition and society to create safe, reliable, efficient and innovative mechanical systems. This has brought about considerable interest for kinematicians and their research work.

Present research in the application of coplanar mechanisms generally falls into one of the following categories:

Multiply Separated Positions or Coplanar Motion	{	(1) Path Generation	{	Position Trajectories Curvature Stationary Curvature
		(2) Function Generation		Input-Output Relative Displacement

Multiple Separated Positions in Coplanar Motion	{	(3) Coupled Mechanisms	{	Single-Link Mechanisms
		(4) Dynamic Systems		Rotational Acceleration Jerk

Figures (3-2...3-7) represent a small portion of the many applications of such mechanisms. These applications are self-evident and require no further elucidation at this point.

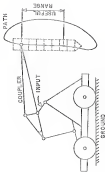


Figure (1-2) Positional Manipulator Employing a Straight-Line Mechanism

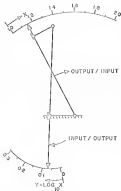


Figure (1-3) Function Generator (Decadal to Log Converter)

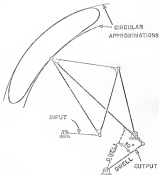


FIGURE C1-10 4-BAR MECHANISM

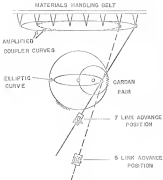


Figure C-10 Coupler Containment Mechanism (Adjustable 9L/9-Side-Deep Formulation)

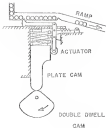


Figure 41-6 Automatic ramp mechanism employing a Double Drum Cam

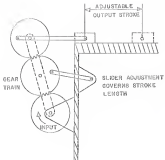


Figure 11-31 Geared Mechanism Telescoped Crank with Adjustable Amplifier

CHAPTER II  
SPHERICAL THEORY  
THE STATE OF THE ART

Fundamentally there is only one constraint function used by the kinematicians to constrain the moving piece. As one would gather from Chapter I, this is the cubic constraint function. The reason is that for convenience kinematicians almost always employ functions which are easily constructed physically and provide utility. These requirements dictate the use of cubic constraint functions. It might be mentioned, however, that one may use any analytic function as a means of constraining the moving piece.<sup>2</sup>

For convenience kinematicians generally separate similar constraints from cubic constraints since the analysis for the latter are significantly more complex. Figures 12-4, 11 represent what will be referred to as the two classes of constraints. In theory the two classes are the same; however, the spherical theory represents a degenerate study of the cubic constraint algorithm. Notice the basic difference in the fixed piece and curvature

<sup>2</sup>The analytic function is a function that can be approximated by a Taylor Series.

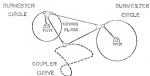


Figure 12-3) Class I Conicoid Constraints)

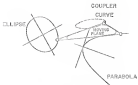


Figure 12-4) Class II Conicoid Constraints)

In Figure (2-22) it will be shown in what follows, the cascade coupler curves are of a higher order, making their versatility somewhat greater.

### Baranovskii Constraints

In solving a problem it is often advantageous to know the number of variables associated with the problem, and the number of parameters specified. Therefore, it becomes imperative to describe the variables and parameters for solving or developing an algorithm of synthesis for a four-bar mechanism (Baranovskii constrained system).

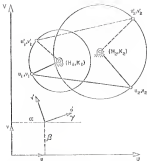
It is seen from Figure (2-3) that there are eight variables associated with two generalized Baranovskii constraint pairs

$$\{(q_1, v_1), (q_2, v_2)\} : \{(q_3, v_3), (q_4, v_4)\}.$$

For a determinate or closed system, the number of variables must equal the number of parameters.<sup>6</sup> From Figure (2-3) each position of the moving plane requires three parameters; hence, it appears that one can only synthesize 3/3 positions. (It is contrary to the Five Position Theory; consequently, one must look further into the problem.

In most analytical systems it becomes desirable to work with revolved parameters. This necessitates of

<sup>6</sup>Common Circular relates a closed system to be a system of revolved and closed art.



NUMBER OF VARIABLES = 2

EACH POSITION REQUIRES 3 PARAMETERS  $(\alpha, \beta, \gamma)$

the system by translating the pole  $P_{11}$  to the origin. Figure 12-41 removes parameters  $a_1$  and  $b_1$ . Rotating the plane and scaling so that  $a_1 = 1$  and  $b_1 = 1$  (Figure 12-41), leaves the following parameters

$$P_{12}, T_{12}, R_{12}, P_{21}, T_{21}, R_{21}, Y_0$$

for the problem specifications. Hence it is seen that

$$\text{number of variables} = \text{number of parameters.}$$

Therefore it is shown that the system becomes a two kinematically closed system i.e., a system which is in agreement with the variable vs. parameters requirements.

A single equation governing all coplanar constraint systems can be expressed as follows:

$$\text{Number of positions} = \frac{\text{number of variables} + 2}{1}.$$

Thus, one can proceed to synthesize Five Linkage Separated Positions in Coplanar Motion.

### Further Theory The Study of Five Linkage Separated Positions in Coplanar Motion

Using the Cartesian frame of reference (orthogonal coordinate system), the transformation of the coordinates of  $A(x, y)$  as a point in the moving plane  $M$ , to the coordinates of the point  $A(x, y)$  of the fixed system  $F$  is given by

$$x = x_{00} + x' \cos \theta + y' \sin \theta \quad (12-12)$$

$$y = y_{00} + x' \sin \theta + y' \cos \theta.$$

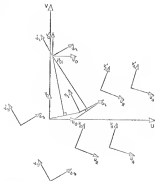


Figure 12-11. Generalized form of the  $v_i$  vectors for the  $3D$  case.

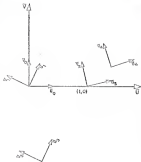


Figure (1-1) rotation and tracking of the Fixed Reference System

From equations (2-1) it is evident that the origin  $o$  of  $EO,VO$  has the coefficients,  $(EO,VO)$  in  $\gamma$ , as seen in Figure (2-5).

The objective of this section is to introduce the seven  $EO,VO$  coefficients which are sufficient to define the values for five prescribed positions (4) constrained by fourier circles. The circles can be in terms of either crank and rocker or slider crank systems. Writing the general form of the circle constraint equation  $Q$  in  $z$  gives

$$Q(E,V) = Q_0(E^2 + V^2) + 2Q_1E + 2Q_2V + Q_3 \quad (2-2)$$

where

$$Q(E,V) = Q_0$$

For a multiply specified position  $i$ , differentiating with respect to the position parameters  $\gamma$  evaluated for a specified  $\gamma_1$  gives

$$Q_1(Q,E,V) = \left. \frac{dQ}{d\gamma} \right|_{\gamma=\gamma_1} = \left. \frac{d}{d\gamma} [Q_0(E^2 + V^2) + 2Q_1E + 2Q_2V + Q_3] \right|_{\gamma=\gamma_1} = 0 \quad (2-3)$$

From equations (2-3), it is seen that  $Q$  and  $V$  are functions of  $\gamma$ , so that

$$Q_1 = \left. \frac{dQ}{d\gamma} \right|_{\gamma=\gamma_1}, \quad V_1 = \left. \frac{dV}{d\gamma} \right|_{\gamma=\gamma_1} \quad (2-4)$$

Writing the derivatives,  $Q_1 = Q_1$ , gives the generalized form for the circle constraint equation as



$$\begin{aligned}
 d_j(z, y) = & \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \gamma_j^2 \right) \psi_j \\
 \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - \gamma_j^2 \right) \psi_j = & \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} - \gamma_j^2 \right) \psi_j = 0.
 \end{aligned}
 \quad (2-3)$$

The coefficients for each position  $j$  become

$$\begin{aligned}
 A_{1j} &= \frac{d^2}{dx^2} (x^2 + y^2) \\
 A_{2j} &= \frac{d^2}{dx^2} (x \cos \gamma_j + y \sin \gamma_j) \\
 A_{3j} &= \frac{d^2}{dx^2} (-x \sin \gamma_j + y \cos \gamma_j) \\
 A_{4j} &= \frac{d^2}{dx^2} (-\cos \gamma_j - 2) \\
 A_{5j} &= \frac{d^2}{dx^2} (-\sin \gamma_j) \\
 A_{6j} &= \frac{d^2}{dx^2} (x) \\
 A_{7j} &= \frac{d^2}{dx^2} (y)
 \end{aligned}
 \quad \gamma_j = \gamma_{1j}, \quad (2-4)$$

The necessary form of the  $A_{ij}$  for all cases of multiply separated positions for up to five positions have been presented in table (2-13). It should be noted that all  $A_{ij}$  are functions of the position parameters,  $x_1, y_1$  and  $\gamma_{1j}$  only.

For this multiply separated positions equation (2-3) can be expressed for the  $j$ th position as

$$\begin{aligned}
 A_{1j} \frac{\partial^2}{\partial x^2} + A_{2j} \frac{\partial^2}{\partial x \partial y} + A_{3j} \frac{\partial^2}{\partial y \partial x} + A_{4j} \frac{\partial^2}{\partial y^2} + A_{5j} \frac{\partial^2}{\partial x} \\
 + A_{6j} \frac{\partial^2}{\partial y} = 0
 \end{aligned}
 \quad (2-5)$$

TABLE II-15. FURTHER COEFFICIENTS FOR THE ADDITIONAL COORDINATES

$k$	$A_{k1}$	$A_{k2}$	$A_{k3}$	$A_{k4}$	$A_{k5}$	$A_{k6}$	$A_{k7}$
0	$\frac{1}{2}(\alpha_1 + \alpha_2)$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2\alpha_3$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$	$\frac{1}{2}(\alpha_1 + \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$
1	$\frac{1}{2}(\alpha_1 - \alpha_2)$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$	$\frac{1}{2}(\alpha_1 - \alpha_2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$
2	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3\alpha_4$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$	$\frac{1}{2}(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$
3	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2\alpha_3$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2\alpha_3\alpha_4$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$	$\frac{1}{2}(\alpha_1\alpha_2 - \alpha_2\alpha_1)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$
4	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2\alpha_3$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2\alpha_3\alpha_4$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$	$\frac{1}{2}(\alpha_1\alpha_2^2 - \alpha_2\alpha_1^2)\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$

For  $k \geq 3$ , the position  $i + k$  is always at the initial position, thus  $L = J = 0$ ,  
 $\alpha_i = \alpha_{i+k} = \alpha_i = 0$ . The first three coordinates with respect to  $\mathbf{q}$ .

linearly independent columns (P-1). The above condition is not true. If  $\mathbf{P}$  is nonsingular,  $\mathbf{P}^{-1}\mathbf{P}$  is the unit diagonal form.

$$\begin{aligned} (\mathbf{A}\mathbf{u}_k + \mathbf{A}_k\mathbf{u} + \mathbf{A}_k\mathbf{v})(\mathbf{u}_k + \mathbf{v}) &= (\mathbf{A}_k) (\mathbf{u}\mathbf{u}_k + \mathbf{v}\mathbf{u}_k) \\ &= (\mathbf{A}\mathbf{u}_k\mathbf{u} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k + \mathbf{U}_k\mathbf{u} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k) = 0 \\ &\quad k = 1, 2, 3, 4 \end{aligned} \quad (10)$$

and

$$\begin{aligned} (\mathbf{A}\mathbf{u}_k + \mathbf{A}_k\mathbf{u} + \mathbf{A}_k\mathbf{v})(\mathbf{u}_k + \mathbf{v}) &= (\mathbf{A}_k) (\mathbf{u}\mathbf{u}_k + \mathbf{v}\mathbf{u}_k) \\ &= (\mathbf{A}\mathbf{u}_k\mathbf{u} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k + \mathbf{A}_k\mathbf{v} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k) = 0 \\ &\quad k = 1, 2, 3, 4 \end{aligned}$$

provided two determinants

$$\begin{vmatrix} (\mathbf{A}\mathbf{u}_k + \mathbf{A}_k\mathbf{u} + \mathbf{A}_k\mathbf{v}) & \mathbf{A}_k & (\mathbf{A}\mathbf{u}_k\mathbf{u} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k) \end{vmatrix} = 0 \quad k = 1, 2, 3, 4$$

and

$$\begin{vmatrix} (\mathbf{A}\mathbf{u}_k + \mathbf{A}_k\mathbf{u} + \mathbf{A}_k\mathbf{v}) & \mathbf{A}_k & (\mathbf{A}\mathbf{u}_k\mathbf{u} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k) & (\mathbf{A}\mathbf{u}_k\mathbf{v} + \mathbf{A}_k\mathbf{u}\mathbf{u}_k) \end{vmatrix} = 0 \quad k = 1, 2, 3, 4$$

both of which must be identically zero.

The two determinants appear to yield characteristic equations that are of third order. Thus is, however, not the case since the determinant provides a nullity of the third order term. Thus these determinants provide two copies of  $\mathbf{u}$  and  $\mathbf{v}$  in the same place which can be expressed as

$$\mathbf{v}_1\mathbf{v}^2 + \mathbf{U}_1\mathbf{u} + \mathbf{v}_1(\mathbf{v} + \mathbf{U}_1\mathbf{u}^2 + \mathbf{U}_1\mathbf{u} + \mathbf{v}_1) = 0 \quad (11)$$

$$\mathbf{v}_2\mathbf{v}^2 + \mathbf{U}_2\mathbf{u} + \mathbf{v}_2(\mathbf{v} + \mathbf{U}_2\mathbf{u}^2 + \mathbf{U}_2\mathbf{u} + \mathbf{v}_2) = 0$$

where

$$\begin{aligned}
 \Gamma_{110} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{111} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{112} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{120} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{121} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{122} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix}
 \end{aligned}$$

and

(2-13)

$$\begin{aligned}
 \Gamma_{211} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{212} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{220} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{221} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{222} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix} + \begin{vmatrix} \bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{22} \end{vmatrix} \\
 \Gamma_{223} &= \begin{vmatrix} \bar{h}_{11} & \bar{h}_{12} \\ \bar{h}_{21} & \bar{h}_{22} \end{vmatrix}
 \end{aligned}$$

The intersections of these circles are the Murnaghan points.

Eliminating  $x$  from equation (2-18) gives the quartic

$$E_0 x^4 + E_1 x^3 + E_2 x^2 + E_3 x + E_4 = 0$$

(2-14)

in  $x$  for the Murnaghan points. The coefficients are

$$E_0 = 6\alpha^2 b^2 - 4\alpha^2 k(a-d) + 4\alpha^2 Z^2$$

$$\begin{aligned}
 E_1 &= 6\alpha^2 d(P_1 a-d) + 4\alpha^2 a d + 6\alpha^2 d(P_1 a-d) \\
 &\quad + 4\alpha^2 d a + 4\alpha^2 d + 2\alpha^2 a d
 \end{aligned}$$

$$\begin{aligned}
 E_2 &= 4\alpha^2 b^2 + 4\alpha^2 d(P_1 a-d) + 4\alpha^2 d(P_1 a-d) + 4\alpha^2 d a \\
 &\quad + 4\alpha^2 d a + 4\alpha^2 d(P_1 a-d) + 4\alpha^2 d a + 4\alpha^2 d a \\
 &\quad + 4\alpha^2 d a
 \end{aligned}$$

$$L_1 = D_1 q_1 (2 f_2 q_2 - 1) + q_1 q_2 (D_2 + 2 q_1 a - 1) + q_1 D_3 - q_1^2 \\ + (q_2 - 1) (2 f_2 q_2 - 1) + q_1 (a - f_2)$$

$$L_2 = (a - q_2^2) + q_2 (2 f_2 - 1) (D_2 - q_2) + q_2 (D_3 - q_2^2) \quad (2-13)$$

where

$$\begin{aligned} a &= T_{11} \cdot T_{11} & f_1 &= T_{11} \cdot T_{12} \\ b &= T_{12} \cdot T_{12} & q &= T_{11} \cdot T_{13} \\ d &= T_{12} \cdot T_{13} & h &= T_{11} \cdot T_{21} \\ e &= T_{12} \cdot T_{21} & p &= T_{11} \cdot T_{22} \\ k &= T_{21} \cdot T_{11} & q &= T_{11} \cdot T_{22} \end{aligned} \quad (2-14)$$

The four solutions to equations (2-12) are  $u_0$ . By removing the term  $v^4$  from equations (2-12), the other pin coordinates,  $v_0$ , is given as

$$v_0 = \frac{(a-f_1) v_0^2 b + (p-f_1) a + (k-e) b}{(a-f_1) v_0 + (b-f_1)} \quad (2-15)$$

The location of the contactpoint in the fixed plane can be related from

$$\begin{bmatrix} P_1 & P_2 \\ Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1/Q_2 \\ Q_1/Q_2 \end{bmatrix} = \begin{bmatrix} -Q_1 \\ -Q_2 \end{bmatrix} \quad (2-16)$$

where

$$\begin{aligned} P_1 &= A_{11} + A_{12} u_0 + A_{13} v_0 \\ P_2 &= A_{21} + A_{22} u_0 + A_{23} v_0 \\ P_3 &= A_{31} + A_{32} u_0 + A_{33} v_0 \end{aligned} \quad (2-17)$$

$$\begin{aligned} i &= 1, 2 \\ n &= 1, 2, 3, 4. \end{aligned}$$

plane for generalized polynomial deg (k) by

$$\begin{aligned} \hat{g}_1 &= -\alpha_1 \alpha_2 \hat{g}_2 \\ \hat{g}_2 &= -\alpha_2 \alpha_3 \hat{g}_3 \\ &\vdots \\ \hat{g}_n &= -\alpha_n \alpha_{n+1} \hat{g}_{n+1} \end{aligned} \quad (2-11)$$

they may be expressed as

$$\hat{g}_1 = \frac{(\alpha_1 \alpha_2 - \alpha_2 \alpha_3)}{(\alpha_1 \alpha_2 - \alpha_2 \alpha_3)} \hat{g}_2 \quad (2-12)$$

$$\hat{g}_2 = \frac{(\alpha_2 \alpha_3 - \alpha_3 \alpha_4)}{(\alpha_2 \alpha_3 - \alpha_3 \alpha_4)} \hat{g}_3 \quad n = 1, 2, 3, 4.$$

It might be mentioned here that this mode of solution is adaptable for arbitrary point synthesis as well as linear point synthesis as illustrated by Figures (2-2,3). Although Figure (2-3) illustrates five positions on the  $U$ -coordinate the quartic (2-12) degenerates for this condition and gives for five positions on the  $V$ -coordinate. This may be verified by observing that for  $\hat{g}_i = 0$ ,  $i = 1, 2, 3, 4$ ,  $\frac{\partial}{\partial \hat{g}_i}$

$$T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24} = 0$$

which causes

$$a, b, c, d, e, f, g = 0$$

and the quartic is vanish. For five positions on the  $U$ -coordinate  $\hat{g}_{12} = 0$ .

$$T_{11}, T_{12}, T_{13}, T_{14}, T_{21}, T_{22}, T_{23}, T_{24} = 0$$

which causes

$$h, i, j, k, l, m, n, o, p, q = 0$$

and the quartic is vanish. A circle passing all the positions about the origin so that  $\hat{x}_1/\hat{x}_{12} = \text{constant}$  reduces this condition and causes only  $P_2$  to vanish in the quartic.

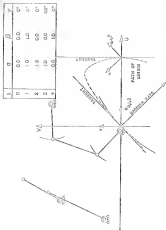


Figure (3-7) Five Mutually Dependent Positions for Case P13-20

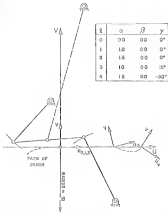


Figure 12-3: Kinematic Synthesis for Case 200-P-2

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Previous work by Tikh [3] improved a earlier approach in locating the fixed points (Hurwitz points) and the singularities (Hurwitz points), while synthesizing nine finite precision points in explicit notation. So that the ideas of visualization of both's work will be subjected to the correcting frame of reference, this work is transformed here into rectangular coordinates.

If from Figure (2-3), one assumes two Hurwitz points exist, a position  $j$  must satisfy the two singular constraints functions following from equation (2-7) as

$$\begin{aligned} \text{Constraint 1: } & \{q_{j1} \sin \gamma_j + q_{j2} \cos \gamma_j = q_{j1} \\ \text{Constraint 2: } & \{d_{j1} \sin \gamma_j + d_{j2} \cos \gamma_j = d_{j1} \end{aligned} \quad (2-20)$$

or

$$\begin{bmatrix} q_{j1} & q_{j2} \\ d_{j1} & d_{j2} \end{bmatrix} \begin{Bmatrix} \sin \gamma_j \\ \cos \gamma_j \end{Bmatrix} = \begin{Bmatrix} q_{j1} \\ d_{j1} \end{Bmatrix} \quad (2-21)$$

where

$$\begin{aligned} q_{j1} &= O_j - R_1 \sin \gamma_j = O_j - R_1 \sin \gamma_j \\ q_{j2} &= O_j - R_2 \sin \gamma_j = O_j - R_2 \sin \gamma_j \\ q_{j3} &= I - R_1 \sin \gamma_j = I - R_1 \sin \gamma_j \\ &= (q_{j1} R_1 + q_{j2} R_2 = \frac{R_1^2 + R_2^2}{2}) \end{aligned} \quad (2-22)$$

and

$$\begin{aligned} \ell_{11} &= l_1^2 + l_2^2 a_1^2 - l_3^2 - a_2^2 r_1^2 \\ \ell_{12} &= l_2^2 - a_2^2 a_1 + l_3^2 - a_2^2 r_2^2 \\ \ell_{13} &= l_1 l_2 - a_2^2 a_1 + l_1 l_3 - a_2^2 r_2^2 \\ &\quad - (a_2^2 x_1 + a_2^2 x_2 - l_3^2) + \frac{a_2^2}{2} l_3 \end{aligned}$$

Using the identity

$$\sin^2 \gamma_1 + \sin^2 \gamma_2 = 1 \quad (2-23)$$

we can substitute and may write the constraints (2-2) in simple position form as

$$\begin{vmatrix} a_{11} & a_{12} \\ \ell_{11} & \ell_{12} \end{vmatrix}^2 + \begin{vmatrix} a_{11} & a_{12} \\ \ell_{11} & \ell_{12} \end{vmatrix}^2 = \begin{vmatrix} a_{21} & a_{22} \\ \ell_{21} & \ell_{22} \end{vmatrix}^2 \quad (2-24)$$

or

$$\begin{aligned} (a_{21} \ell_{21} - a_{22} \ell_{22})^2 + (a_{21} a_{22} - \ell_{21} \ell_{22})^2 = \\ (a_{11} \ell_{11} - \ell_{12} a_{12})^2. \end{aligned} \quad (2-25)$$

From equation (2-25) one can write the positional matrix

for positions  $j = 0, 1, 2, 3, 4, 5, 6, 7, 8$  as

$$\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \begin{pmatrix} a_1^2 \\ a_1^2 r_1^2 \\ a_1^2 r_2^2 \\ a_2^2 \\ a_2^2 r_1^2 \\ a_2^2 r_2^2 \\ l_1^2 \\ l_1^2 r_1^2 \\ l_1^2 r_2^2 \end{pmatrix} = 0 \quad (2-26)$$

$$a_{jm} = (a) \begin{Bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ a_{4j} \\ a_{5j} \\ a_{6j} \\ a_{7j} \\ a_{8j} \end{Bmatrix} = 0 \quad (2-27)$$

and

$$a_{jm} = f(x_1, x_2, x_3, x_4, x_5, x_6) \\ j = 1, 2, 3, 4, 5, 6, 7, 8.$$

From equations (2-26,27) it is obvious that there are eight unknowns

$$\{(a_1, v_1), (a_2, v_2)\} \cup \{(a_1, v_1), (a_2, v_2)\}.$$

Thus one can write the eight precision point equations and theoretically satisfy his criterion for nine finitely separated precision points.

CHAPTER III  
LINKS: GEOMETRY  
AND STATE OF THE ART

Early works by Burdick [14], Singer [15] and Waldman [16] have provided the kinematician with the capability of using matrix computations for determining displacements of the moving plane. Later works by Rongue and Freudenstein [17] and Woo and Freudenstein [17] provided an extension of the earlier authors' works that had employed infinitesimal displacements. A recent publication by Freudenstein, Bottoms and Weinstein [18] upgrades the kinematician with a concise method of synthesizing via finite positions of the moving plane. This paper also treats all degenerate cases of the design for finite synthesis through the position theory.

The intention here is to formulate a generalized method for synthesizing any multiply separated positions of the moving plane by defining the circle motion point curve. It is also the author's objective to present a new algorithm for synthesizing any multiply separated positions of the moving plane. This algorithm produces two quantities  $h$  and  $r$  which relate the 18 scale points for curve constraints.

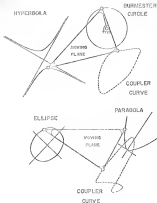


Figure 45-11 Typical Gear Constraints

### Circle Theory Thirteen $\theta_1$ Parameters

While synthesizing a circle constraint pair, we do not advantage to analyze the number of variables vs. the number of parameters for the system. It is seen from Figure G-10 that there are fourteen variables associated with two generalized circle constraint pairs:

$$\begin{aligned} & (\alpha_1, \sigma_1), (\theta_1, \delta_1), \left(\frac{a}{r}\right)_1, \left(\frac{b}{r}\right)_1, (\theta_1, 1) \\ & (\alpha_2, \sigma_2), (\theta_2, \delta_2), \left(\frac{a}{r}\right)_2, \left(\frac{b}{r}\right)_2, (\theta_2, 1). \end{aligned}$$

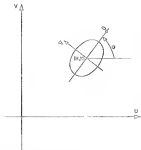
After normalizing and removing  $\alpha_1, \delta_1, \alpha_2$  and  $\delta_2$  by translation, rotating and scaling, as outlined for the translation constraints, the number of synthesizable positions becomes

$$\text{number of positions} = \frac{\text{number of variables} + 1}{2}$$

$$\text{number of positions} = \frac{14 + 1}{2}$$

$$\text{number of positions} = 7.$$

The maximum number of positions (coordinates and angles) which can be synthesized using circle constraints becomes seven. Hence one can proceed to synthesize seven multiply separated positions in Copeland routine for circle constraint pairs.



EQUATIONS OF TRANSFORMATION  
(TRANSLATION AND ROTATION)

$$U = (V - K)\sin\theta + (U - H)\cos\theta$$

$$V = (V - K)\cos\theta - (U - H)\sin\theta$$

Figure 3-3) Properties of a Circle in the Pinned Plane

The equation for a generalizing ellipse with its center in the origin can be expressed as:

$$x_1^2 + x_2^2 + x_3^2 + x_4 = 0 \quad (3-1)$$

where  $x_1, x_2, x_3, x_4$  are defined for the ellipse as illustrated in Table [3-1]. If the center of the ellipse is not at the origin, the inverse transformation

$$\begin{aligned} \bar{X} &= (X - X_0)\cos\theta + (Y - Y_0)\sin\theta \\ \bar{Y} &= (Y - Y_0)\cos\theta - (X - X_0)\sin\theta \end{aligned} \quad (3-2)$$

gives the new coordinates as shown in Figure [3-2] for an ellipse. Substituting equations (3-2) into (3-1) gives

$$\begin{aligned} x_1[(X^2 - 2X + X_0^2)\cos^2\theta + 2(Y - Y_0 + X_0\sin\theta)\cos\theta \\ + (Y^2 - 2Y + Y_0^2)\sin^2\theta] + \\ x_2[(X^2 - 2X + X_0^2)\cos^2\theta - 2(Y - Y_0 + X_0\sin\theta)\cos\theta \\ + (Y^2 - 2Y + Y_0^2)\sin^2\theta] + \\ x_3[(Y - Y_0)\cos\theta - (X - X_0)\sin\theta] + x_4 = 0. \end{aligned} \quad (3-3)$$

Rearranging and collecting terms gives

$$\begin{aligned} &X^2(x_1\cos^2\theta + x_2\sin^2\theta) + XY[2(Y_0 - X_0)\sin\theta\cos\theta] \\ &+ Y^2[-2X_0\sin\theta\cos\theta + x_2\sin^2\theta] + 2X(x_1 - x_2)\sin\theta\cos\theta + x_3\sin\theta \\ &+ Y^2(x_1\sin^2\theta + x_2\cos^2\theta) \quad (3-4) \\ &+ Y[-2X_0\sin\theta\cos\theta + x_2\cos^2\theta] + 2X(x_1 - x_2)\sin\theta\cos\theta + x_3\cos\theta \\ &+ [2X^2(x_1\cos^2\theta + x_2\sin^2\theta) + 2Y^2(x_1\sin^2\theta + x_2\cos^2\theta) + \\ &2XY(x_1 - x_2)\sin\theta\cos\theta + x_3(X\sin\theta - Y\cos\theta) + x_4] = 0. \end{aligned}$$

CIRCLE	$x_2 = 0$ $\frac{x_1}{x_2} = \frac{x_3}{x_4} = \frac{1}{\frac{1}{R^2}}$	$L_1^2 - 4L_2 \neq 0$
ELLIPSE	$x_2 \neq 0$ $\frac{x_1}{x_2} = \frac{1}{\frac{1}{R^2}}$ $\frac{x_3}{x_4} = \frac{1}{\frac{1}{R^2}}$	$L_1^2 - 4L_2 \neq 0$ $\frac{x_1}{x_2}$ (POSITIVE)
HYPERBOLA	$x_2 \neq 0$ $\frac{x_1}{x_2} = \frac{1}{\frac{1}{R^2}}$ $\frac{x_3}{x_4} = \frac{1}{\frac{1}{R^2}}$	$L_1^2 - 4L_2 \neq 0$ $\frac{x_1}{x_2}$ (NEGATIVE)
PARABOLA	$x_2 = 0$ $x_4 \neq 0$ $\frac{x_3}{x_1} = -\frac{2a}{x_1}$	$L_1^2 - 4L_2 = 0$
a. RADII OF THE CIRCLE b. a. SEM-MAJ b. SEM-AXIS	2P. VERTICES TO FOCUS DISTANCE FOR THE PARABOLA $L_1, L_2$ TAKEN FROM EQUATIONS (2-22)	

A point limit position is given by the limit values obtained from the origin of the limit plane, where  $x_1 = \theta_1$  and  $y_1 = 0$  and the transformation equations

$$\begin{aligned} X &= x_1 \cos \theta + y_1 \sin \theta + a \\ Y &= x_1 \sin \theta + y_1 \cos \theta + b \end{aligned} \quad (2-1)$$

where

$$X = x_0$$

$$Y = y_0$$

Let  $x_1, y_1$  be a point of

$$\begin{aligned} &[x^2(x_1 \cos^2 \theta + x_1 \sin^2 \theta) + x^2(x_1 \sin^2 \theta + x_1 \cos^2 \theta) + \\ &2xy(x_1 - x_1) \sin \theta \cos \theta + y^2(\sin^2 \theta - \cos^2 \theta) + y^2] \end{aligned}$$

from the circle constraint equations (2-4) to give

$$\begin{aligned} 0 &= x^2[(x_1 \cos^2 \theta + x_1 \sin^2 \theta) + \\ 0 &= xy[2(x_1 - x_1) \sin \theta \cos \theta] + \\ 0 &= y^2[-2x_1 \sin^2 \theta + x_1 \sin^2 \theta - 2x_1 \sin \theta \cos \theta - x_1 \sin \theta + \\ 0 &= x^2[(x_1 \sin^2 \theta + x_1 \cos^2 \theta) + \\ 0 &= y^2[-2x_1 \sin^2 \theta + x_1 \cos^2 \theta - 2x_1 \sin \theta \cos \theta \\ &+ x_1 \cos \theta] = 0. \end{aligned} \quad (2-5)$$

Substituting the transformation equations (2-5) provides

$$\begin{aligned} X &= x_1 \cos \theta + y_1 \sin \theta + a \\ Y &= x_1 \sin \theta + y_1 \cos \theta + b \end{aligned} \quad (2-6)$$

and making the following substitutions

$$\begin{aligned} x_1 &= x_1 \cos^2 \theta + x_1 \sin^2 \theta \\ x_2 &= 2(x_1 - x_1) \sin \theta \cos \theta \\ x_3 &= -2x_1 \sin^2 \theta + 2x_1 \sin \theta \end{aligned}$$

$$\begin{aligned}x_1 &= R_1 \sin^2 \theta + R_2 \cos^2 \theta \\x_2 &= -R_1 x_3 - \sin x_3 + R_2 \cos \theta\end{aligned}$$

(3-14)

allows one to write the six position stress constraint vectors as

$$(\sigma)_i = 0 \quad (3-15)$$

or

$$\begin{bmatrix} (x_1^2 - x_1^0) & (x_1 y_1 - x_1^0 y) & (x_1 - x) & (x_1^2 - x^2) & (x_1 - x) \\ (x_1^2 - x^0) & 0 & 0 & 0 & 0 \\ (x_1^2 - x^0) & 0 & 0 & 0 & 0 \\ (x_1^2 - x^0) & 0 & 0 & 0 & 0 \\ (x_1^2 - x^0) & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = 0. \quad (3-16)$$

Substituting the transformation equations (2-2) into the first typical equation of (3-16) will give

$$\begin{aligned}(a_{11})(x^2 - x^0) + 2a_{12}(xy) + 2a_{13}(x) + 2a_{14}(y) + a_{15} &= \\ (a_{12})(x^2 - x^0) + 2a_{13}(x) + (a_{14} + a_{15})(y) + (a_{12} - a_{14}) & \\ (y) + a_{15} &= 2x_1 + \\ (a_{13})(x) + a_{14}(y) + a_{15} &= \\ (-a_{14})(x^2 - x^0) + 2a_{12}(xy) + 2a_{13}(x) + 2a_{14}(y) + a_{15} &= \\ (a_{13})(x) + a_{14}(y) + a_{15} &= 0\end{aligned}$$

$$j = 1, 2, 3, 4, 5.$$

(3-17)

For any multiple repeated position a typical equation can be written as above by taking the coefficients  $a_{ij}$  from Table (3-2). For a finite position the coefficients

TABLE (3-13) SERIES COEFFICIENTS FOR THE CASE OF CONCENTRATED

$n$	$\delta_{00}$	$\delta_{01}$	$\delta_{02}$	$\delta_{03}$	$\delta_{04}$	$\delta_{05}$	$\delta_{06}$	$\delta_{07}$	$\delta_{08}$	$\delta_{09}$	$\delta_{10}$	$\delta_{11}$	$\delta_{12}$	$\delta_{13}$
0	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1	0.999	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.998	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.997	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.996	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.995	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.994	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.993	0.007	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.992	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.991	0.009	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.990	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

NOTE: 1. The values of  $\delta_{ij}$  are given in the table above.  
2. The values of  $\delta_{ij}$  are given in the table above.

are given by

$$\begin{aligned}
 a_{13} &= -\sin^2 \gamma_j \\
 a_{23} &= \sin \gamma_j \cos \gamma_j \\
 a_{33} &= \cos^2 \gamma_j - 1 \\
 a_{43} &= \sin \gamma_j \\
 a_{12} &= a_j \cos \gamma_j \\
 a_{22} &= a_j \sin \gamma_j \\
 a_{32} &= b_j \cos \gamma_j \\
 a_{42} &= b_j \sin \gamma_j \\
 a_{53} &= a_j^2 \\
 a_{11} &= a_j b_j \\
 a_{21} &= a_j \\
 a_{31} &= b_j^2 \\
 a_{41} &= b_j
 \end{aligned}
 \tag{1-12}$$

which may be expressed singularly in the coefficients of equations (1-4). From the typical equation one can develop the Six Position Curve Constraint Matrix for six mutually separated positions as defined by positions 1 ~ 6, 1, 2, 3, 4, 5 of the moving plane.

#### The Six Position Curve Function SIX CASE

As shown by Freudenstein, Rothbart and Hestenes the Curve Section Point Curve is of seventh degree. This is verified from equations (1-12) by adding column 1 to column 4, which gives for the typical equation after subscripting for  $i$ th position.

$$\begin{aligned}
 & (B_{11}(\alpha^2 - \alpha^2) - 2B_{11}(\alpha\alpha) + 2B_{11}(\alpha)) - 2B_{11}(\alpha\alpha) + B_{11}(\alpha, \alpha) \\
 & (B_{11}(\alpha^2 - \alpha^2) + 2B_{11}(\alpha\alpha) + 2B_{11}(\alpha, \alpha) - 2B_{11}(\alpha) + 2B_{11} - B_{11}) - 2B_{11}(\alpha, \alpha) \\
 & (B_{11}(\alpha) - B_{11}(\alpha) + B_{11}(\alpha, \alpha) \\
 & (2(B_{11} + B_{11})(\alpha) + 2(B_{11} - B_{11})(\alpha) + (B_{11} + B_{11})(\alpha, \alpha) \\
 & (B_{11}(\alpha) + B_{11}(\alpha) + B_{11}(\alpha, \alpha) = 0 \quad i = 1, 2, 3, 4, 5.
 \end{aligned}
 \tag{2-13}$$

The cubic constraint matrix then becomes

$$(C)(x) = 0 \tag{2-14}$$

where

$$\begin{aligned}
 C_{11} &= B_{11}(\alpha^2 - \alpha^2) - 2B_{11}(\alpha\alpha) + 2B_{11}(\alpha) - 2B_{11}(\alpha) + B_{11} \\
 C_{11} &= B_{11}(\alpha^2 - \alpha^2) + 2B_{11}(\alpha\alpha) + 2B_{11} + B_{11}(\alpha) + 2B_{11} - B_{11}(\alpha \\
 & \quad + B_{11} \\
 C_{11} &= B_{11}(\alpha) - B_{11}(\alpha) + B_{11} \\
 C_{11} &= 2(B_{11} + B_{11})(\alpha) + 2(B_{11} - B_{11})(\alpha) + (B_{11} + B_{11}) \\
 C_{11} &= B_{11}(\alpha) + B_{11}(\alpha) + B_{11} \\
 i &= 1, 2, 3, 4, 5.
 \end{aligned}
 \tag{2-15}$$

The expansion of the determinant of the cubic constraint matrix [C] can best be accomplished by performing all minors. Observing the last three columns and comparing these coefficients with those described in [3] for four position Denavit-Hartenberg Theory and dividing column 4 by 2 it is seen that the coefficients are the same.<sup>2</sup> This allows for

<sup>2</sup>The generalized four position Denavit-Hartenberg Theory is formulated in Appendix [4].

is justified) can be expressed (under certain  $1 \leq n \leq 5$ ) in terms of the generalized cubic form Bernstein Theory. Setting the minor for positions 3, 4 and 5 be  $Q_1$ , 2, 4, and 1, be  $Q_2$ , etc., there will be ten minors  $Q_n$  for the  $(3 \times 4)$ 's of the last three columns. The minors for the first two columns are given by a quartic of  $u$  and  $v$  and may be expressed as

$$\begin{aligned} & q_{j,n}(u^4 - v^4)^2 + 4(u^4 - v^4)uv + 4(uv)^2 + \\ & (q_{j,n}u^4 + q_{j,n}v^4)(u^4 - v^4) + \\ & (q_{j,n}u^4 + q_{j,n}v^4 + q_{j,n}) (uv) + \\ & q_{j,n}(u^4) + q_{j,n}(v^4) + \\ & (q_{j,n}u^4 + q_{j,n}v^4 + q_{j,n}) \end{aligned} \quad (3-14)$$

$$n = 1, 2, 3, \dots, 10$$

where

$$\begin{aligned} q_{1,n} &= (q_{1,1}q_{1,n} - q_{1,n}q_{1,1}) \\ q_{2,n} &= (q_{1,2}(q_{1,n} + q_{1,n}) + 2(q_{1,2}q_{1,n} - q_{1,n}q_{1,2}) \\ &\quad - q_{1,n}(q_{1,2} + q_{1,2})) \\ q_{3,n} &= (q_{1,3}(q_{1,n} - q_{1,n}) - 2(q_{1,3}q_{1,n} - q_{1,n}q_{1,3}) \\ &\quad - q_{1,n}(q_{1,3} - q_{1,3})) \\ q_{4,n} &= (q_{1,4}(q_{1,n} + q_{1,n}) - 2(q_{1,4}(q_{1,n} + q_{1,n}) \\ &\quad + q_{1,n}(q_{1,4} + q_{1,4}))) \\ q_{5,n} &= (q_{1,5}(q_{1,n} - q_{1,n}q_{1,n}) - 2(q_{1,5}(q_{1,n} - q_{1,n}) \\ &\quad + q_{1,n}(q_{1,5} - q_{1,5}))) \\ q_{6,n} &= (q_{1,6}(q_{1,n} - q_{1,n}q_{1,n} + q_{1,n}(q_{1,n} - q_{1,n}) \\ &\quad - q_{1,n}(q_{1,n} + q_{1,n}))) \\ &\quad - 2(q_{1,6}(q_{1,n} - q_{1,n}q_{1,n}) + q_{1,n}(q_{1,n} - q_{1,n}) \\ &\quad - q_{1,n}(q_{1,n} - q_{1,n}))) \end{aligned} \quad (3-17)$$

$$\begin{aligned}
\hat{g}_{10} &= [2(a_{11}^2 a_{10} + a_{10}^2 a_{10}) - (a_{10}^2 a_{10} + a_{10}^2 a_{10}) \\
&\quad + 2(a_{10}^2 (a_{10} + a_{10}) - a_{10}^2 (a_{10} + a_{10}))] \\
\hat{g}_{11} &= [(a_{11}^2 a_{10} + a_{10}^2 a_{10}) - (a_{10}^2 a_{10} + a_{10}^2 a_{10}) \\
&\quad + 2(a_{10}^2 (a_{10} - a_{10}) - a_{10}^2 (a_{10} - a_{10}))] \\
\hat{g}_{12} &= [2(a_{11}^2 a_{10} - a_{10}^2 a_{10}) \\
&\quad + a_{10}^2 (a_{10} + a_{10}) - a_{10}^2 (a_{10} + a_{10})] \\
\hat{g}_{13} &= [2(a_{10}^2 a_{10} - a_{10}^2 a_{10}) \\
&\quad + a_{10}^2 (a_{10} - a_{10}) - a_{10}^2 (a_{10} - a_{10})] \\
\hat{g}_{14} &= (a_{10}^2 a_{10} - a_{10}^2 a_{10})
\end{aligned}$$

where  $a_{ij} = a_{ij}$

$$i = 1, 2, 3, \dots, 10.$$

The expansion of the determinant  $|G|$  expressed in terms of the products of minors becomes

$$\begin{aligned}
\frac{1}{n-1} &(-1)^{n-1} a_{10} (a_{10}^2 (a^2 - v^2)^2 + 2(a^2 - v^2)(av + a(av)^2) \\
&\quad + (a_{10}^2 + a_{10}^2)(a^2 - v^2) \\
&\quad + (a_{10}^2 + a_{10}^2 + a_{10}^2)(av) \\
&\quad + a_{10}(a^2) + a_{10}(a^2) \\
&\quad + (a_{10}^2 + a_{10}^2 + a_{10}^2) = 0
\end{aligned}
\tag{1-10}$$

where

$$n = (1 + 2 + 1_2 + 1_2 + \dots, 10)$$

and the signs are taken from Appendix (A).

### Verification of Circularity

The authors Frobenius, Schur, and Fowler reviewed a knowledge of trichromaticity of the cone vision point normal however, they give no verification of this

where  $\gamma = \frac{1}{2}(\alpha + \beta)$ . With Appendix C-1 (2) (3) where  $\alpha$  and  $\beta$  are replaced by

$$\begin{aligned} & (\alpha_1 \alpha^2 + \alpha_2 \alpha^2)(x^2 + y^2) + \alpha_3 \alpha^2 + \\ & \alpha_4 \alpha^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha^2 + \alpha_7 \alpha^2 + \alpha_8 \alpha^2 = 0 \end{aligned} \quad (2-10)$$

By substituting the first term of equation (2-10) the algebraic order of the circle section point curve becomes

$$\begin{aligned} & \lambda_1 x + \lambda_2 y (x^2 + y^2) [(x^2 + y^2)^2 + \\ & 2xy(x^2 - y^2) + xy] \end{aligned} \quad (2-11)$$

where  $\lambda_1$  and  $\lambda_2$  are linear functions on the coefficients  $\alpha_{ij}$ . It can be shown by substituting

$$x = r \cos \theta$$

into equation (2-11) and letting  $r \rightarrow 0$  that the resulting slope  $\theta$  is related by

$$(x^2 + y^2)^2 (\lambda_1 x + \lambda_2 y) = 0 \quad (2-12)$$

which will give six imaginary asymptotes indicating trilinearity.

### The Describing Parameters on the Section Curve

For a particular  $\alpha$  and  $\beta$  the circle section point curve (2) becomes a circle. The application of Goursat's Rule provides for a particular  $\alpha$  and  $\beta$

$$\begin{aligned}
\hat{L}_1 &= L_1 \hat{L}_2 \\
\hat{M}_1 &= L_1 \hat{M}_2 \\
\hat{N}_1 &= L_1 \hat{N}_2 \\
\hat{K}_1 &= L_1 \hat{K}_2
\end{aligned}
\quad (3-27)$$

Substituting equations (3-25a) and (3-27) into (3-24) gives

$$\begin{bmatrix} (L_1 \cos^2 \theta - 2 \sin \theta \cos \theta) & (L_1 \sin^2 \theta + 2 \sin \theta \cos \theta) \\ (L_1 \cos^2 \theta - \sin^2 \theta) & (L_1 \cos^2 \theta - \sin^2 \theta) \end{bmatrix} \begin{Bmatrix} \hat{L}_2 \\ \hat{M}_2 \end{Bmatrix} = 0. \quad (3-28)$$

By adding column 1 to 2 and dividing 1 by  $\cos^2 \theta$  the system reduces to

$$\begin{bmatrix} L_1 - 2 \tan \theta & L_1 + 1 \\ L_1 - \tan^2 \theta & L_1 - 1 \end{bmatrix} = 0.$$

Subtracting column 2 from column 1 yields the following form

$$\begin{bmatrix} 1 - 2 \tan \theta & L_1 + 1 \\ 1 - \tan^2 \theta & L_1 - 1 \end{bmatrix} = 0. \quad (3-29)$$

The determinant, or the angle of rotation function, becomes

$$1 - \tan^2 \theta - 2(L_1 - 1) \tan \theta - L_1 = 0. \quad (3-30)$$

The centerpoint coordinates are given by equations (3-8a) and (3-8b) for  $\theta_p/\theta_0 = 0$  as taken from Table

(1-1) for the above,  $u, v, w$  and  $z$  are given

$$\begin{bmatrix} -2k_1 & -k_1 \\ -k_1 & -2k_1 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}.$$

Substitution of equations (1-23) gives

$$\begin{bmatrix} -2k_1 & -k_1 \\ -k_1 & -2k_1 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad (1-24)$$

which allows for expressing the centerpoint coordinates as

$$u = \frac{k_1 z_1}{k_1^2 + k_1} = \frac{k_1 z_1}{k_1(k_1 + 1)}$$

and (1-25)

$$v = \frac{k_1 z_2}{k_1^2 + k_1} = \frac{k_1 z_2}{k_1(k_1 + 1)}.$$

For  $k = 0$  and  $k_1/k_2 = 0$ , the constrained equation (1-9) becomes

$$\begin{aligned} T_1 / (Ox^2 - Oy^2 + R^2 \sin^2 \theta) + 2(uv - uz - v^2 + w)(\sin \theta \cos \theta) \\ + (u^2 - 2uv + R^2 \cos^2 \theta) + \\ T_2 / (Ox^2 - Oy^2 + R^2 \cos^2 \theta) - 2(uv - uz - v^2 + w)(\sin \theta \cos \theta) \\ + (u^2 - 2uv + R^2 \sin^2 \theta) + T_3 = 0 \end{aligned}$$

letting

$$\begin{aligned} T_1 &= Ox^2 - Oy^2 + R^2 \sin^2 \theta + 2(uv - uz - v^2 + w)(\sin \theta \cos \theta) \\ &\quad + (u^2 - 2uv + R^2 \cos^2 \theta) \\ T_2 &= Ox^2 - Oy^2 + R^2 \cos^2 \theta - 2(uv - uz - v^2 + w)(\sin \theta \cos \theta) \\ &\quad + (u^2 - 2uv + R^2 \sin^2 \theta) \end{aligned}$$

and combining Eqs. (10) and (11) gives the system

$$\begin{bmatrix} -\tau_1 & \tau_1 \\ L_1 \cos^2 \theta - \sin^2 \theta & L_1 \sin^2 \theta - \cos^2 \theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} -x \\ y \end{Bmatrix} \quad (12a)$$

The inversion provides the semi-axis

$$\begin{aligned} \frac{x_1}{\tau_1} &= \frac{-L_1 \sin^2 \theta - \cos^2 \theta}{\tau_1 [L_1 \sin^2 \theta - \cos^2 \theta] - \tau_1 [L_1 \cos^2 \theta - \sin^2 \theta]} \\ \frac{y_1}{\tau_1} &= \frac{[L_1 \cos^2 \theta - \sin^2 \theta]}{\tau_1 [L_1 \sin^2 \theta - \cos^2 \theta] - \tau_1 [L_1 \cos^2 \theta - \sin^2 \theta]} \end{aligned} \quad (12b)$$

All of the properties for conic constraint synthesis of the multiply separated positions can now be expressed for the ellipse, ellipse and hyperbola. As will be shown in the following section, the parabola requires a different algorithm in locating the center and angle of rotation for the semi-axis. The geometric center of the parabola is as follows which requires that

$$L_1^{-1} - 4L_2 = 0.$$

A method of relating the center to the vertex is derived and illustrated for this special case.

#### The Describing Parameters of the Parabola

For a particular  $u$  and  $v$  the conic constraint matrix [C] becomes defined and the application of Cayley's Rule provides

$$\begin{aligned}
 \bar{u}_1 &= \bar{u}_1 \bar{u}_1 \\
 \bar{u}_2 &= \bar{u}_1 \bar{u}_2 \\
 \bar{u}_3 &= \bar{u}_1 \bar{u}_3 \\
 \bar{u}_4 &= \bar{u}_1 \bar{u}_4
 \end{aligned}
 \quad (3-31)$$

(2) The particular set of coordinates  $(x, y)$  from equation (3-1) and Table (3-1) if the condition

$$u_1^2 + 4u_2 = 0 \quad (3-32)$$

is satisfied then the conic is a parabola and Table (3-1) suggests that

$$\begin{aligned}
 u_1 &= 0 \\
 u_2 &= 0.
 \end{aligned}$$

For a parabola equations (3-6) become

$$\begin{aligned}
 x_1 &= x_1 \cos^2 \theta \\
 x_2 &= 2x_1 \sin \theta \cos \theta \\
 x_3 &= -2x_1 \sin \theta \cos \theta + x_1 \sin^2 \theta \\
 x_4 &= x_1 \sin^2 \theta \\
 x_5 &= -2x_1 \sin \theta \cos \theta + x_1 \cos^2 \theta.
 \end{aligned}
 \quad (3-33)$$

This gives for equations (3-33a) and (3-33e) the forms

$$2 \cos^2 \theta - x_1 \cos^2 \theta = 0$$

and

$$\sin^2 \theta - x_1 \sin^2 \theta = 0$$

both satisfying equation (3-31). The angle of the transformation can be expressed as

$$\gamma_{00} = \epsilon_0 \quad (3-13)$$

It will be shown later that the value of  $\epsilon_0/\epsilon_1$  determines the direction which the parabola opens and establishes the passing of  $\beta$ . Equations (3-13a) and (3-17a) can be grouped to form the characteristic set

$$\begin{bmatrix} 2\epsilon_1 & \epsilon_0 \\ \epsilon_1 & 2\epsilon_0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} -\epsilon_1 + \epsilon_0 \sin \theta \\ -\epsilon_1 + \epsilon_0 \cos \theta \end{Bmatrix} \quad (3-14)$$

The discriminant of equations (3-14) can be expressed by

$$\Delta_1^2 = 4\epsilon_1\epsilon_0$$

or

$$\Delta_1^2 = \Delta_0 = 0$$

as defined by equation (3-11). If equations (3-14) are to satisfy the condition of the center, they must be coincident. The coincidence also requires the discriminant

$$\begin{bmatrix} \epsilon_1 & (-\epsilon_1 + \epsilon_0 \sin \theta) \\ 2\epsilon_1 & (-\epsilon_1 + \epsilon_0 \cos \theta) \end{bmatrix} = 0 \quad (3-15)$$

to be zero. The discriminant gives the  $\epsilon_0/\epsilon_1$

$$\frac{\epsilon_0}{\epsilon_1} = \frac{\cos \theta (1 - \epsilon_1 \sin \theta)}{\epsilon_1 (1 - \epsilon_1)} \quad (3-16)$$

$$\begin{aligned}
\dot{R}_{1,1} &= (-2\alpha R_1 + \alpha R_1) = 0 \\
\dot{R}_{1,2} &= 2\alpha R_1 + 2\alpha R_1 = 0 \\
\dot{R}_{1,3} &= (-\alpha R_1 + 2\alpha R_1) = 0 \\
\dot{R}_{1,4} &= 0 \\
\dot{R}_{2,1} &= 0 \\
\dot{R}_{2,2} &= 0 \\
\dot{R}_{2,3} &= 0 \\
\dot{R}_{2,4} &= 0 \\
\dot{R}_{3,1} &= 0 \\
\dot{R}_{3,2} &= 0
\end{aligned}
\tag{3-42}$$

$$= 1, 1, 1, 1, 1, 0$$

which comes from equation (3-11). For matrix conditions of the moving plane the system can be simplified, by applying Cramer's Rule, as

$$\begin{pmatrix}
(2\alpha^2 - \alpha^2)(R_1 - R_1) + 2\alpha R_1 \\
(2\alpha^2 - \alpha^2)(R_2) - 2\alpha(R_1 - R_1) \\
\alpha R_1 + \alpha R_1 \\
-\alpha R_1 + \alpha R_1 \\
R_1 \\
R_1
\end{pmatrix} = (2\alpha) \begin{pmatrix}
2\alpha R_1 + \alpha R_1 \\
-2\alpha R_1 + \alpha R_1 \\
\alpha R_1 + 2\alpha R_1 \\
-\alpha R_1 + 2\alpha R_1 \\
R_1 \\
R_1
\end{pmatrix}
\tag{3-43}$$

where  $\alpha_{ij}$  are functions of the  $R_{ij}$  coefficients.

The first two expressions of equation (3-43) are functions of only  $R_1$ ,  $R_2$ ,  $R_3$  and may be expressed as

$$\begin{aligned}
(2\alpha^2 - \alpha^2) &= 2\alpha_{1,1}R_1 + \alpha_{1,2}R_2 + \alpha_{1,3}R_3 + \\
(2\alpha R_1 - 2\alpha_{1,1}R_1 + \alpha_{1,2}R_2 - 2\alpha_{1,3}R_3) &= \alpha_{1,4}R_4 + \\
(-2\alpha^2 - \alpha^2) &= \alpha_{2,1}R_1 + \alpha_{2,2}R_2 + \alpha_{2,3}R_3 = 0
\end{aligned}$$

and

$$(3-44)$$

$$\begin{aligned}
(1-2\alpha_1 + 2\alpha_{12}x + \alpha_{11}x^2 - \alpha_{13})x_1 + \\
(2\alpha_2 - \alpha_3) - (2\alpha_{12} + \alpha_{11})x + (2\alpha_{13} - \alpha_{11})x^2 - 2\alpha_{11}x_1x_2 = 0 \\
(2\alpha_3 - \alpha_{11}x + \alpha_{12}x^2 - \alpha_{13})x_2 = 0.
\end{aligned}
\quad (3-43)$$

Though these two equations were directly derived from the typical or generalized set, they must satisfy the characteristic set and the characteristic equation. By multiplying equation (3-43) by  $x_{1k}$  and subtracting from the  $k$ th equation (3-11) and by multiplying equation (3-43) by  $x_{2k}$  and subtracting from the  $k$ th equation (3-12) gives for the  $k$ th or typical equation,

$$\begin{aligned}
& (2\alpha_{12}x_k + \alpha_{11}x_k^2 + \alpha_{13})x_{1k} + \\
& (2\alpha_2x_k + \alpha_{11}x_k + \alpha_{13})x_{2k} = \\
& (2\alpha_3x_k + \alpha_{11}x_k + \alpha_{13})x_{1k} + \\
& (2\alpha_{12}x_k + \alpha_{11}x_k^2 + \alpha_{13})x_{2k} = \\
& (2\alpha_{12}x_k + \alpha_{11}x_k^2 + \alpha_{13})x_{1k} = 0
\end{aligned}
\quad (3-44)$$

i = 0, 1, 2, 3, 4

where

$$\begin{aligned}
\alpha_{11} &= 2(2\alpha_{11} + 3\alpha_{12}x_{11} + 3\alpha_{13}x_{11}) \\
\alpha_{12} &= -(2\alpha_{11} + 3\alpha_{12}x_{11} + 3\alpha_{13}x_{11}) \\
\alpha_{13} &= -(2\alpha_{11} + 3\alpha_{12}x_{11} + 3\alpha_{13}x_{11}) \\
\alpha_{14} &= (2\alpha_{11} + 3\alpha_{12} + 3\alpha_{13}(\alpha_{11} + \alpha_{12}) + 3\alpha_{13}(\alpha_{11} + \alpha_{12})) \\
\alpha_{15} &= (2\alpha_{11} - 3\alpha_{12} + 3\alpha_{13}(\alpha_{11} - \alpha_{12}) + 3\alpha_{13}(\alpha_{11} - \alpha_{12})) \\
\alpha_{16} &= (2\alpha_{12} + 3\alpha_{12}x_{11} + 3\alpha_{13}x_{11}) \\
\alpha_{17} &= (2\alpha_{12} + 3\alpha_{12}x_{11} + 3\alpha_{13}x_{11}) \\
\alpha_{18} &= (2\alpha_{12} - 3\alpha_{12}x_{11} - 3\alpha_{13}x_{11})
\end{aligned}
\quad (3-45)$$

$$\begin{aligned}
 c_{11g} &= b_{11g} \\
 c_{12g} &= b_{12g} \\
 c_{13g} &= b_{13g} \\
 c_{14g} &= b_{14g}
 \end{aligned}
 \quad g = 2, 3, 4, 5, 6.$$

Since that columns 4 and 5, the coefficients of  $b_4$  and  $b_5$ , are not altered by this manipulation, employing the technique outlined for the Five Position Harmonic Theory for compound determinants with linear orthogonal or linear parallel elements gives

$$\begin{aligned}
 & (c_{14g}u + c_{15g}v + c_{16g})x_1 + (c_{24g}u + c_{25g}v + c_{26g})x_2 + \\
 & (c_{34g}u + c_{35g}v + c_{36g})x_3 + (c_{44g}u + c_{45g}v + c_{46g})x_4 + \\
 & (-c_{14g}v + c_{15g})x_5 + (c_{24g}u + c_{25g})x_6 = 0 \quad g = 2, 3, 4, 5, 6
 \end{aligned}$$

and

$$\begin{aligned}
 & (c_{14g}u + c_{15g}v + c_{16g})x_1 + (c_{24g}u + c_{25g}v + c_{26g})x_2 + \\
 & (c_{34g}u + c_{35g}v + c_{36g})x_3 + (c_{44g}u + c_{45g}v + c_{46g})x_4 + \\
 & (-c_{14g}v + c_{15g})x_5 + (c_{24g}u + c_{25g})x_6 = 0 \quad g = 2, 3, 4, 5, 6.
 \end{aligned}$$

The two determinants (characteristic equations) of the square matrices provide two equations in  $u$  and  $v$  relating the 14 harmonic points for the general case. The equation can be expressed as

$$\begin{aligned}
 & h_{11}u^4 + (h_{12} + h_{13}v)u^3 + (h_{14} + h_{15}v + h_{16}v^2)u^2 + \\
 & (h_{21} + h_{22}v + h_{23}v^2 + h_{24}v^3)u +
 \end{aligned}$$

$$(b_{111} + b_{112}x + b_{113}x^2 + b_{114}x^3 + b_{115}x^4) = 0 \quad (12-51)$$

and

$$\begin{aligned} & (b_{121}x^5 + (b_{122} + b_{123}x)x^4 + (b_{124} + b_{125}x + b_{126}x^2)x^3 + \\ & (b_{127} + b_{128}x + b_{129}x^2 + b_{130}x^3)x^2 + \\ & (b_{131} + b_{132}x + b_{133}x^2 + b_{134}x^3 + b_{135}x^4) = 0 \end{aligned} \quad (12-52)$$

where the  $b_{pq}$ 's are the expansions of the respective determinants. The  $b_{pq}$ 's can be expressed as

$$\begin{aligned} b_{11} &= \begin{vmatrix} x & x_1 & x_2 & x_3 & x_{12} & x_{13} & x_{14} \\ x_1 & x_2 & x_3 & x_4 & x_{12} & x_{13} & x_{14} \\ x_2 & x_3 & x_4 & x_5 & x_{12} & x_{13} & x_{14} \\ x_3 & x_4 & x_5 & x_6 & x_{12} & x_{13} & x_{14} \\ x_4 & x_5 & x_6 & x_7 & x_{12} & x_{13} & x_{14} \\ x_5 & x_6 & x_7 & x_8 & x_{12} & x_{13} & x_{14} \\ x_6 & x_7 & x_8 & x_9 & x_{12} & x_{13} & x_{14} \end{vmatrix} \\ b_{12} &= \begin{vmatrix} x & x_1 & x_2 & x_3 & x_{12} & x_{13} & x_{14} \\ x_1 & x_2 & x_3 & x_4 & x_{12} & x_{13} & x_{14} \\ x_2 & x_3 & x_4 & x_5 & x_{12} & x_{13} & x_{14} \\ x_3 & x_4 & x_5 & x_6 & x_{12} & x_{13} & x_{14} \\ x_4 & x_5 & x_6 & x_7 & x_{12} & x_{13} & x_{14} \\ x_5 & x_6 & x_7 & x_8 & x_{12} & x_{13} & x_{14} \\ x_6 & x_7 & x_8 & x_9 & x_{12} & x_{13} & x_{14} \end{vmatrix} \\ & \vdots \\ & \vdots \\ & \vdots \\ b_{13} &= \frac{1}{x} \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_{12} & x_{13} & x_{14} \\ x_2 & x_3 & x_4 & x_5 & x_{12} & x_{13} & x_{14} \\ x_3 & x_4 & x_5 & x_6 & x_{12} & x_{13} & x_{14} \\ x_4 & x_5 & x_6 & x_7 & x_{12} & x_{13} & x_{14} \\ x_5 & x_6 & x_7 & x_8 & x_{12} & x_{13} & x_{14} \\ x_6 & x_7 & x_8 & x_9 & x_{12} & x_{13} & x_{14} \end{vmatrix} \end{aligned} \quad (12-53)$$

and similarly for equation (12-52). Thus the quartics defining the 16 Veronese points can be related by a closed form technique.

The intersections of these quartics can be found by setting

$$\begin{aligned} d_{m1} &= d_{m0} \\ d_{m2} &= d_{m0} + d_{m0}x \\ d_{m3} &= d_{m0} + d_{m0}x + d_{m0}x^2 \end{aligned}$$

$$\begin{aligned}
 d_{20} &= d_{201} + d_{202}x^2 + d_{203}x^4 + d_{204}x^6 \\
 d_{21} &= d_{211} + d_{212}x^2 + d_{213}x^4 + d_{214}x^6 + \\
 &\quad d_{215}x^8 + d_{216}x^{10}
 \end{aligned}$$

$$x = 1, 2$$

(3-43)

and employing Rytov's Diaphanous effect of distinction to give the resultant  $S$  for  $u_0$  as

$$S(u_0) = \begin{vmatrix}
 d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & 0 & 0 & 0 \\
 0 & d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & 0 & 0 \\
 0 & 0 & d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & 0 \\
 0 & 0 & 0 & d_{41} & d_{42} & d_{43} & d_{44} & d_{45} \\
 d_{51} & d_{52} & d_{53} & d_{54} & d_{55} & 0 & 0 & 0 \\
 0 & d_{61} & d_{62} & d_{63} & d_{64} & d_{65} & 0 & 0 \\
 0 & 0 & d_{71} & d_{72} & d_{73} & d_{74} & d_{75} & 0 \\
 0 & 0 & 0 & d_{81} & d_{82} & d_{83} & d_{84} & d_{85}
 \end{vmatrix} = 0$$

For the real Hermite points the resultant  $S(u_0)$  must be singular. Therefore various numerical methods of minimizing  $S$  may be employed to obtain the real Hermite points (zeros of  $S$ ).

The detouring parameters of the curve may be found using the same technique outlined for six position synthesis equations (3-22,...,3-31) for each of the intermediates of the quantities (Hermite points).

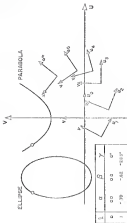


Figure 3-20 A Conic Coordinates Plot Analyzing Array Position at the Moving Plane

#### CHAPTER IV

##### ANALYSIS OF THE CONTROLLING CONSTRAINT PAIR

As given by Vander and Freudenstein [10], synthesis is not yet found not be restricted to classical (Ducasse's) mechanisms. In general, any some constrained set provides a means of constraining the moving plane which is the requirement of synthesis. This is to say, and will be verified in what follows, that the kinematic constraints are merely convenient basic mechanisms which utilize simplified modes of synthesis and analysis. Since the researchers Vander and Freudenstein [11], Freudenstein, Bottoms, and Shuster [12] and thus earlier have algorithms for synthesizing positions in coplanar motion for single constraints, it seems only logical that an algorithm of analysis is necessary. Therefore it is the author's intent to introduce such an analysis algorithm for any combination of the following cases

Circle  
Ellipse  
Hyperbola  
Parabola.

From equation (3-6) and equation (3-8) the basic constraint equation was given as

$$\begin{aligned} (x^2 - x^2_0) + (y^2 - y^2_0) + 2x_0x + 2y_0y + c_0 &= 0 \\ (x^2 - x^2_0) + (y^2 - y^2_0) + 2x_0x + 2y_0y + c_0 &= 0 \end{aligned} \quad (4-6)$$

where

$$\begin{aligned}U &= u_{0000} + v_{0000} + 2 \\V &= u_{0001} + v_{0001} + 2\end{aligned}\quad (3-62)$$

and

$$\begin{aligned}X_1 &= X_1 \cos^2 \theta + X_2 \sin^2 \theta \\X_3 &= 2(X_1 - X_2) \sin \theta \cos \theta \\X_4 &= -2EX_1 - EX_2 - 2X_3 \sin \theta \\X_5 &= X_1 \sin^2 \theta + X_2 \cos^2 \theta \\X_6 &= -EX_1 - 2EX_2 + 2X_3 \cos \theta.\end{aligned}\quad (3-63)$$

For a known conicoid set the variables

$$u, v, \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}, E, X_5, \theta$$

become known.\* Rearranging gives the quadratic

$$d_1 u^2 + (d_2 u + d_3) v + d_4 v^2 + d_5 v + d_6 = 0 \quad (3-64)$$

where

$$\begin{aligned}d_1 &= E_1 \\d_2 &= E_2 \\d_3 &= \sin \theta (EX_1 + EX_2) + \cos \theta (EX_1 + EX_2) + E_3 \\d_4 &= E_4 \\d_5 &= \sin \theta (EX_1 + EX_2) + \cos \theta (EX_1 + EX_2) + E_5 \\d_6 &= \sin^2 \theta (u^2 - v^2) (X_1 - X_2) + \sin \theta (X_1 - X_2) + \\&\quad \sin \theta \cos \theta (u^2 - v^2) (X_1 - X_2) + \cos \theta (X_1 - X_2) +\end{aligned}$$

---

\*For the ellipse, ellipse and hyperbola  $E_1 = 0$  and for the parabola  $E_1 = E_2$ .

$$\begin{aligned} \max_{\mathbf{u}} &= \mathbf{u}^T (\mathbf{A}_1 \mathbf{X}_1 + \mathbf{b}_1 \mathbf{X}_2) \\ \max_{\mathbf{u}} &= -\mathbf{u}^T \mathbf{X}_1 + \mathbf{u}^T \mathbf{X}_2 \end{aligned}$$

(14-5)

When all two unknowns, sets are known

$$\text{Conventional Pair} \begin{cases} \mathbf{u}_1, \mathbf{v}_1, \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}_1, \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}_2, \mathbf{u}_2, \mathbf{X}_3, \mathbf{u}_4 \\ \mathbf{u}_1, \mathbf{v}_2, \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}_1, \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}_2, \mathbf{u}_1, \mathbf{X}_3, \mathbf{u}_4 \end{cases}$$

equations (4-3) may be summarized as follows

$$[\mathbf{u}_{m+1}]$$

and

$$[\mathbf{u}_{m+2}]$$

$$\begin{aligned} m &= 1, 2 \\ n &= 1, 2, 3, 4, 5, 6 \end{aligned}$$

The two unknown sets provide two quadratics for equations (4-5) and may be represented as

$$\begin{aligned} \text{Position} & \begin{bmatrix} \mathbf{u}_{m+1} & (\mathbf{A}_{m+1} \mathbf{X} + \mathbf{b}_{m+1}) \\ \mathbf{u}_{m+2} & (\mathbf{A}_{m+2} \mathbf{X} + \mathbf{b}_{m+2}) \end{bmatrix} \begin{Bmatrix} \mathbf{u}^2 \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} -(\mathbf{A}_{m+1} \mathbf{X}^2 + \mathbf{A}_{m+1} \mathbf{X} + \mathbf{b}_{m+1}) \\ -(\mathbf{A}_{m+2} \mathbf{X}^2 + \mathbf{A}_{m+2} \mathbf{X} + \mathbf{b}_{m+2}) \end{Bmatrix} \\ \text{Rotation} & \end{aligned} \quad (4-6)$$

in terms of  $\mathbf{X}$  for the moving plane. Applying Cramer's Rule to equation (4-6) equating the second row and equating to the first row gives the quadratic for  $\mathbf{X}$

$$\mathbf{X}_1 \mathbf{X}^2 + \mathbf{X}_2 \mathbf{X}^2 + \mathbf{X}_3 \mathbf{X}^2 + \mathbf{X}_4 \mathbf{X} + \mathbf{X}_5 = 0 \quad (4-7)$$

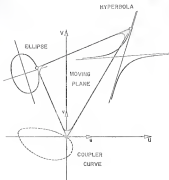


Figure 04-13 Coupler Curve Function for Gear Tooth Interference

in terms of  $\gamma$  of the rolling plane. The roots of the double provide large  $\beta$  coordinates of the rolling  $P$ - $Q$  joint angle  $\gamma$  provided. The corresponding  $\alpha$  is taken from the inversion of equations (4-6). The coefficients  $E_i$  are defined by equations (2-13) and (2-14).

### Higher-Order Constraints

It is often advantageous to analyze the geometric velocity, acceleration, jerk, jounce, etc., of a particular constraint pair. This is accomplished by differentiating the position functions equations (4-6) with respect to  $\gamma$  to give

$$\begin{array}{l} \text{Geometric} \\ \text{Velocity} \\ \text{Matrices} \end{array} \begin{bmatrix} \dot{x}_{11} & \dot{x}_{12} \\ \dot{x}_{21} & \dot{x}_{22} \end{bmatrix} \begin{Bmatrix} \dot{\alpha} \\ \dot{\beta} \end{Bmatrix} = - \begin{Bmatrix} \dot{x}_{13} \\ \dot{x}_{23} \end{Bmatrix} \quad (4-8)$$

where

$$\begin{aligned} \dot{x}_{m1} &= \dot{x}_{m0}\alpha + \dot{x}_{m2}\beta + \dot{x}_{m3} \\ \dot{x}_{m2} &= \dot{x}_{m1}\alpha + \dot{x}_{m2}\beta + \dot{x}_{m3} \\ \dot{x}_{m3} &= \dot{x}_{m1}\alpha + \dot{x}_{m2}\beta + \dot{x}_{m3} \end{aligned} \quad (4-9)$$

Differentiating of the Velocity Matrices will give the accelerating

$$\begin{array}{l} \text{Geometric} \\ \text{Acceleration} \\ \text{Matrices} \end{array} \begin{bmatrix} \ddot{x}_{11} & \ddot{x}_{12} \\ \ddot{x}_{21} & \ddot{x}_{22} \end{bmatrix} \begin{Bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{Bmatrix} = \begin{Bmatrix} \ddot{x}_{13} \\ \ddot{x}_{23} \end{Bmatrix} \quad (4-10)$$

where

$$\gamma_{1i} = -(\partial_{1i}^{1'} a + \partial_{1i}^{2'} \dot{b} + \partial_{1i}^{3'} \ddot{c}) \quad i = 1, 2,$$

Differentiating equation (4-10) provides the jerk

$$\begin{array}{ll} \text{Geometric} & \begin{bmatrix} \dot{c}_{11} & \dot{c}_{12} \end{bmatrix} \begin{Bmatrix} a^{1'1'} \\ b^{1'2'} \end{Bmatrix} = \begin{Bmatrix} \dot{g}_{11} \\ \dot{g}_{12} \end{Bmatrix} \\ \text{Jerk} & \\ \text{Matrix} & \begin{bmatrix} \dot{c}_{11} & \dot{c}_{12} \end{bmatrix} \begin{Bmatrix} a^{1'1'} \\ b^{1'2'} \end{Bmatrix} = \begin{Bmatrix} \dot{g}_{11} \\ \dot{g}_{12} \end{Bmatrix} \quad (4-11) \end{array}$$

where

$$\dot{g}_{1i} = -(\partial_{1i}^{1'1'} a' + \partial_{1i}^{2'1'} a'' + \partial_{1i}^{3'1'} \dot{a}' + \partial_{1i}^{1'2'} \dot{b}' + \partial_{1i}^{2'2'} \dot{b}'') \quad i = 1, 2,$$

Differentiating eqn. (4-11) provides for the yojack

$$\begin{array}{ll} \text{Geometric} & \begin{bmatrix} \ddot{c}_{11} & \ddot{c}_{12} \end{bmatrix} \begin{Bmatrix} a^{1'1''} \\ b^{1'2''} \end{Bmatrix} = \begin{Bmatrix} \ddot{g}_{11} \\ \ddot{g}_{12} \end{Bmatrix} \\ \text{Yojack} & \\ \text{Matrix} & \begin{bmatrix} \ddot{c}_{11} & \ddot{c}_{12} \end{bmatrix} \begin{Bmatrix} a^{1'1''} \\ b^{1'2''} \end{Bmatrix} = \begin{Bmatrix} \ddot{g}_{11} \\ \ddot{g}_{12} \end{Bmatrix} \quad (4-12) \end{array}$$

where

$$\ddot{g}_{1i} = -(\partial_{1i}^{1'1'1'} a' + \partial_{1i}^{2'1'1'} a'' + \partial_{1i}^{3'1'1'} a''' + \partial_{1i}^{1'2'1'} \dot{a}' + \partial_{1i}^{2'2'1'} \dot{a}'' + \partial_{1i}^{1'2'2'} \dot{b}' + \partial_{1i}^{2'2'2'} \dot{b}''').$$

While illustrating that differentiating and applying Cramer's Rule provide a means of analyzing all orders of contacts, note, however that the higher order contacts are dependent on the lower order contacts.

for identifying  $\hat{h}_i$  according to equation (4-1) for the derivatives  $f_{\text{eq}}$ . This formulation allows for ease in computer programming and mathematical calculations.

### Analysis of Warmerter Constraints

As stated earlier in this text, Warmerter Theory consists of a degenerate form of Gauss Theory, thus it becomes appropriate to investigate the consequences of the analysis algorithm. From equation (4-5) it is seen that for Warmerter constraints

$$\hat{h}_{11} = \hat{h}_{12} = \hat{h}_{13} = \hat{h}_{14}$$

and

$$\hat{h}_{21} = \hat{h}_{22} = 1.0$$

Subtracting equation (8-2a) from equation (8-2b) provides

$$\hat{h}_{11}x^2 + \hat{h}_{12}x + \hat{h}_{13}x^2 + \hat{h}_{14}x + \hat{h}_{15} = 0$$

$$(\hat{h}_{11} - \hat{h}_{13})x^2 + (\hat{h}_{12} - \hat{h}_{14})x + (\hat{h}_{15} - \hat{h}_{15}) = 0$$

and the linear dependence of  $x$  and  $x^2$  provides a quadratic rather than a quartic in terms of the moving plane coordinates.

The above relation points out that the order of the coupled matrix are different. Thus one would expect

\*\*\*\*\*

\*For circular constraints  $\hat{h}_1 = \hat{h}_2$  and  $\hat{h} = \hat{h}^2$ . These conditions require that  $\hat{h}_1 = \hat{h}_2$  and  $\hat{h}_3 = 0$  for both constrained pairs.

1000

[illegible]



greater variability in synthesis using single representations as has been proven to be the case.

Locating the Pole of  
the Firing Plane

The pole or instant center is the position in the firing plane which has an instantaneous velocity of zero. And since the pole coordinates are given by the known-formulas:

$$\begin{aligned}x_p &= v_{y2} \cos \gamma - v_{y1} \sin \gamma + x \\v_p &= v_{y2} \cos \gamma + v_{y1} \sin \gamma + v\end{aligned}$$

Differentiating with respect to  $\gamma$  gives for  $0_p' = v_p' = 0$

$$\begin{aligned}0 &= v_{y2} \sin \gamma + v_{y1} \cos \gamma \\0 &= -v_{y2} \cos \gamma + v_{y1} \sin \gamma.\end{aligned}\tag{4-13}$$

Substituting equations (4-11) into equations (4-13) provides

$$\begin{bmatrix} (f_{11} \sin \gamma - f_{22} \cos \gamma) & (f_{12} \cos \gamma + f_{21} \sin \gamma) \\ (f_{11} \cos \gamma + f_{22} \sin \gamma) & (f_{12} \sin \gamma - f_{21} \cos \gamma) \end{bmatrix} \begin{bmatrix} v_p \\ \omega_p \end{bmatrix} = \begin{bmatrix} f_{13} \\ f_{23} \end{bmatrix}.\tag{4-14}$$

Applying Cramer's Rule gives the pole's position in the firing plane for any  $\gamma$  chosen. From the pole position the location of the deflection circle, ball point and other higher order properties can be determined.

### Location of Arbitrary Points in the Working Plane

For arbitrary points in the working plane, one simply applies the transformation

$$\text{Arbitrary Transformation} \begin{cases} \bar{U} = U_{\text{orig}} - U_{\text{orig}} + u \\ \bar{V} = V_{\text{orig}} + V_{\text{orig}} + v \end{cases} \quad (4-13)$$

where  $\bar{U}$  and  $\bar{V}$  are the coordinates of the arbitrary point. Equation (4-13) shows that the system's algorithm is not altered since  $\bar{U}$  and  $\bar{V}$  are uniquely determined by the transformation.

As for the geometric velocity, one must differentiate, with respect to  $\tau$ , to give

$$\text{Geometric Velocity Transformation} \begin{cases} \bar{U}' = -\dot{U}_{\text{orig}} - \dot{U}_{\text{orig}} + u' \\ \bar{V}' = \dot{U}_{\text{orig}} - \dot{V}_{\text{orig}} + v' \end{cases} \quad (4-14)$$

Differentiation of the velocity transformation gives for the geometric acceleration

$$\text{Geometric Acceleration Transformation} \begin{cases} \bar{U}'' = -\ddot{U}_{\text{orig}} + \ddot{U}_{\text{orig}} + u'' \\ \bar{V}'' = -\ddot{U}_{\text{orig}} - \ddot{V}_{\text{orig}} + v'' \end{cases} \quad (4-15)$$

and further differentiation gives the jerk as

$$\text{Geometric Jerk Transformation} \begin{cases} \bar{U}''' = \ddot{U}_{\text{orig}} + \ddot{U}_{\text{orig}} + u''' \\ \bar{V}''' = -\ddot{U}_{\text{orig}} + \ddot{V}_{\text{orig}} + v''' \end{cases} \quad (4-16)$$

The above equations show that if the geometric position, velocity, acceleration, jerk, etc., of the origin are known, then these properties determine the position.



$$\begin{bmatrix} [X_2 - X_1 \sin \theta] \\ X_1 \end{bmatrix} = \begin{bmatrix} [X_1 + X_2 \cos \theta] \\ [X_2] \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} -[X_1] \\ -[X_1 + X_2 \cos \theta] \end{Bmatrix} \quad (3-42)$$

The inversion of the equations provides

$$x = \frac{-[X_2 X_1 - (X_1 + X_2 \cos \theta) (X_1 + X_2 \cos \theta)]}{[X_1 (X_1 + X_2 \sin \theta) - X_2 (X_1 + X_2 \sin \theta)]}$$

and

$$y = \frac{[X_2 - X_1 \sin \theta] (X_1 + X_2 \cos \theta) + [X_1 X_2]}{[X_1 (X_1 + X_2 \sin \theta) - X_2 (X_1 + X_2 \sin \theta)]} \quad .$$

Simplifying gives

$$x = \frac{1}{2} \left\{ \frac{\left[ \frac{X_2}{X_1} \right]^2 - [1] \cos^2 \theta - \frac{[X_2]}{[X_1]} \cos \theta}{[X_1 \sin \theta + \cos \theta] \left[ \frac{X_1}{X_2} \right] - [1] \cos^2 \theta - \cos \theta \left[ \frac{X_2}{X_1} \right]} \right\}$$

and

$$y = \frac{1}{2} \left\{ \frac{\left[ \frac{X_2}{X_1} \right]^2 + [1] X_2 \cos \theta + X_2 \sin \theta \left[ \frac{X_2}{X_1} \right] - [1] X_1 \cos^2 \theta - X_1 \left[ \frac{X_2}{X_1} \right]}{[X_1 \sin \theta + \cos \theta] \left[ \frac{X_1}{X_2} \right] - [1] \cos^2 \theta - \cos \theta \left[ \frac{X_2}{X_1} \right]} \right\} \quad (3-43)$$

where

$$\left[ \frac{X_2}{X_1} \right] = (x^2 + 2xX_1 + xX_1 + x^2X_1 + xX_1) \quad .$$

The above coordinates define the position of the vertex for the specific parabola corresponding to the set  $\{u, v\}$  and complete the description of the parabola in the fixed plane.

In the previous section it was shown that the equations of the moving plane provide a seventh degree polynomial in terms of  $u$  and  $v$  (coordinates of the axis points) in the moving plane. The addition of a seventh position will provide a second seventh degree polynomial in  $u$  and  $v$  indicating that there are 49 intersections of the two curves. As shown by Friedenstein, Kottens and Kotteler, of these 49 theoretical solutions one must subtract 18 for the circular points at infinity, 18 for the common poles and 1 for the common null points, leaving 18 parameter points for the general case.\*

To obtain the determinants of the  $5 \times 5$  matrix in terms of  $u$  and  $v$  is a formidable task. Therefore it becomes desirable to express the matrix in some other format. If the system is arranged in terms of common coefficients, the generalized equation of motion (hybrid equation) can be expressed as

$$\begin{aligned} & \mathbf{R}_{11} [1u^4 - v^4] \mathbf{C}_1 = \mathbf{R}_{11} + \text{Im}(\mathbf{R}_{11}) + \\ & \mathbf{R}_{12} [1u^4 - v^4] \mathbf{C}_2 = \text{Im}(\mathbf{R}_{11} - \mathbf{R}_{12}) + \\ & \mathbf{R}_{13} [u\mathbf{C}_1 + v\mathbf{C}_2] + \\ & \mathbf{R}_{14} [-v\mathbf{C}_1 + u\mathbf{C}_2] + \\ & \mathbf{R}_{15} [2uv\mathbf{C}_1 + v^2\mathbf{C}_2] + \end{aligned}$$

---

\*The singularities at infinity are a result of the function being bilinear as previously shown in this text.

velocity, acceleration, jerk, etc., of any point in the moving plane.

### Conversion to Real Time

It should be noted that conversion to real time requires that differentiations be taken with respect to time and not the angle  $\gamma$ . The absolute velocity can be expressed as

$$\begin{aligned} \text{Absolute} \\ \text{Time} \\ \text{Velocity} \end{aligned} \begin{cases} \frac{d^2}{dt^2} = i_1 \frac{d^2}{d\gamma^2} + \frac{d\omega}{dt} \\ \frac{d^2}{dt^2} = i_2 \frac{d^2}{d\gamma^2} + \frac{d\omega}{dt} \end{cases} \quad (4-13)$$

where

$$i_1 = -\bar{C} \sin \gamma + \bar{V} \cos \gamma$$

$$i_2 = \bar{C} \cos \gamma + \bar{V} \sin \gamma.$$

For the absolute acceleration one simply differentiates equation (4-13) to obtain

$$\begin{aligned} \text{Absolute} \\ \text{Time} \\ \text{Acceleration} \end{aligned} \begin{cases} \frac{d^3}{dt^3} = i_1 \frac{d^3}{d\gamma^3} - i_1 \frac{d^2 \omega}{dt^2} + \frac{d^3 \omega}{dt^3} \\ \frac{d^3}{dt^3} = i_2 \frac{d^3}{d\gamma^3} + i_2 \frac{d^2 \omega}{dt^2} + \frac{d^3 \omega}{dt^3} \end{cases} \quad (4-14)$$

and similarly for higher order motions.

Reverting equations (4-13,14) it is seen that the relations

$$\frac{d^2}{dt^2} = \frac{d^2}{d\gamma^2} + \frac{d^2 \omega}{dt^2}$$

(3) required in the algorithm in real time. If continuously assuming the origin as the circumplex of constant or defined  $\omega$  (rate of angular rotation) then

$$\frac{dx}{dt} = R_{\omega} \cos \alpha$$

$$\frac{dy}{dt} = R_{\omega} \sin \alpha$$

$$\frac{dz}{dt} = -\left[ \frac{R_{\omega}}{R_p} \right] z$$

where

$R_{\omega}$  = centerpoint to circumplex length

$R_p$  = circumplex to pole length.

For higher order derivatives differentiation must be taken with respect to time for  $\alpha$ ,  $R$ ,  $\gamma$ ,  $\omega$ ,  $R_{\omega}$  and  $R_p$ .

# CHAPTER V

## LINKAGE SYNTHESIS FOR THE GENERALIZED CONIC CONSTRAINT SET

Often the kinematician must have the ability to synthesize a conic constraint set while satisfying coordinate positions (precision points) in explicit motion. This requires factoring the angle  $\tau$  of the moving plane or the absolute angles of the derivatives.

From Chapter III the quantified conic constraint equation was given by

$$\begin{aligned} (x^2 - x^2)x_1 + 2(x^2 - x^2)x_1 + 2(x^2 - x^2)x_2 \\ + (y^2 - y^2)x_3 + 2(y^2 - y^2)x_4 = 0 \end{aligned} \quad (5-1)$$

where  $x$  and  $y$  are defined by equations (3-7) and  $x_1$  by equation (3-8). Specifying the following parameters

$$\begin{aligned} \text{Constraint Set} & \left\{ \begin{aligned} & \text{and} \\ & \text{fraction point} \end{aligned} \right. \left\{ \begin{aligned} & \phi, \tau, R, E, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \phi, \alpha, \beta \end{aligned} \right. \end{aligned}$$

and letting each position carry the subscript  $i$  will give for the finite case

$$\begin{aligned} a_{1i} \sin^2_{\phi_1} + a_{1i} \sin \phi_1 \cos \phi_1 + a_{1i} \sin \tau_1 + a_{1i} \cos \tau_1 + a_{1i} = 0 \\ i = 1, 2, 3, 4, 5, 6 \end{aligned} \quad (5-2)$$

a transcendental equation of fourth degree where

$$\begin{aligned}a_{4,1} &= -(x_1 - x_2)(u^2 - v^2) - 2ux_1 \\a_{3,1} &= (x_1)(u^2 - v^2) - 2(x_1 - x_2)uv \\a_{2,1} &= (u_1)(ux_1 - 2vx_1) + u_2(vx_1 - 2ux_1) + (ux_1 + vx_1) \\a_{1,1} &= (u_1)(2ux_1 + vx_1) + u_2(ux_1 + 2vx_1) + (ux_1 + vx_1) \\a_{0,1} &= (u_1^2x_1 + u_1u_2x_2 + u_2^2x_3 + u_1^2x_4 + u_2^2x_5 - (ux_1 + vx_1))\end{aligned}\quad (1-9)$$

Differentiating and equating provides

$$a_{4,1}\sin^2\gamma_1 + a_{3,1}\sin^2\gamma_2 + a_{2,1}\sin^2\gamma_3 + a_{1,1}\sin\gamma_4 + a_{0,1} = 0 \quad (1-10)$$

where

$$\begin{aligned}a_{4,1} &= u_{1,1}^2 + u_{2,1}^2 \\a_{3,1} &= 2(u_{1,1}u_{2,1} + u_{1,1}u_{3,1}) \\a_{2,1} &= u_{1,1}^2 + u_{2,1}^2 - u_{3,1}^2 + 2u_{1,1}u_{4,1} \\a_{1,1} &= 2(u_{1,1}u_{3,1} - u_{1,1}u_{4,1}) \\a_{0,1} &= u_{1,1}^2 - u_{3,1}^2\end{aligned}\quad (1-11)$$

Equation (1-10) has a closed form solution resulting in four angles for each position or  $(4)^3$  three-valued sets of relations satisfying the seven precision points.

In prescribing undetermined displacements it will be shown that there is a unique  $\phi^h$  and  $\psi^h$  for the relative change prescribed. Differentiating equation (1-3) will give

$$\dot{\phi}_{1,1}\phi^h + \dot{\phi}_{2,1}\psi^h + \dot{\phi}_{3,1} = 0 \quad (1-12)$$

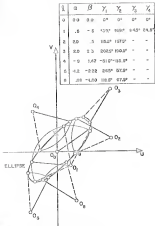


Figure (B-12) could not determine while analyzing  
 Binary Prediction Model

found the  $k_{pg}$  in eq. (20b). One finds that in specifying infinitesimal positions the constraints are specified only the relative change of the coefficients by

$$b_{11}d^1 + b_{12}d^2 = 0, \quad (5-8)$$

increasing equations (5-6) and (5-7) as a linear set for the first derivative gives

$$\begin{aligned} (D_{11}b_{11}d^1 + D_{12}b_{12}d^2) &= -D_{11}b_{11} \\ b_{11}d^1 + b_{12}d^2 &= 0, \end{aligned} \quad (5-9)$$

Applying Cramer's Rule provides the proper  $d^1, d^2$  for the order desired synthesis.<sup>8</sup>

For higher order synthesis one need only differentiate equation (5-6) further as given by equations (4-18, 11, 12) take the coefficients  $k_{pg}$  from table (4-1) and proceed as outlined.

### Linkage Synthesis for the Burmeister Coupling Pair

As one might gather from the previous chapters, Burmeister pair synthesis is an elementary study of coupler section constraint synthesis. From equation (3-4) the Burmeister constraint equation is given by

$$Q^2 - q^2 - r^2 - w^2q_0 = 2(2 - w)q_1 + 2Q - wq_2 = 0$$

---

<sup>8</sup>These are the unique absolute values to be used in the synthesis algorithm.

where  $Q$  and  $r$  are given from Eqs. (12-11) and  $Q_0/Q_1$  from equation (12-16). By specifying the following parameters

$$\begin{aligned} \text{Generation Pair} \\ \text{and} \\ \text{Fecundity Point} \end{aligned} \left\{ \begin{aligned} &(\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2) \end{aligned} \right.$$

and letting each parameter carry the subscript (1) gives for the finite case

$$a_{11} \sin \gamma + a_{12} \cos \gamma + a_{13} = 0 \quad (5-4)$$

which is a transcendental equation of second degree in  $\gamma$ —the coefficients are expressed by

$$\begin{aligned} a_{11} &= Q_1 \cos \gamma_0 \alpha - Q_0 \alpha - RQ \\ a_{12} &= Q_1 \cos \gamma_0 \beta + Q_0 \beta - RQ \\ a_{13} &= Q_1 + RQ \alpha + Q_0 + RQ \beta \\ &\quad - (Q_1 R + R_1 R - Q_1 \frac{R^2}{2} + \frac{R_1 R^2}{2}) \end{aligned} \quad (5-5)$$

Rearranging and squaring provides the quadratic

$$a_{14} \sin^2 \gamma + a_{15} \sin \gamma + a_{16} = 0 \quad (5-6)$$

where

$$\begin{aligned} a_{14} &= a_{11}^2 + a_{12}^2 \\ a_{15} &= 2a_{11}a_{12} \\ a_{16} &= a_{11}^2 + a_{12}^2 \end{aligned} \quad (5-7)$$

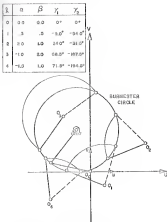


Figure (1-8) Crankshaft Link Synthesis While Satisfying Five Precision Points

The derivatives of  $\alpha^k$  and  $\beta^k$  also encompass the necessary constraints for the relative change procedure. Differentiating equation (3-9) gives for the first order infinitesimal case

$$[E_{11}]_1 \alpha' + [E_{12}]_1 \beta' + [E_{13}]_1 = 0 \quad (3-13)$$

where the  $E_{pq}$  coefficients are taken from Table (4-1).

In synthesizing infinitesimal positions we specify only the relative change of the link lengths

$$h_{12} \alpha' + h_{13} \beta' = 0, \quad (3-14)$$

combining equations (3-13) and (3-14) gives the linear system

$$\begin{aligned} [E_{11}]_1 \alpha' + [E_{12}]_1 \beta' &= -[E_{13}]_1 \\ h_{12} \alpha' + h_{13} \beta' &= 0. \end{aligned} \quad (3-15)$$

Applying Cramer's Rule provides the proper  $\alpha'$  and  $\beta'$  for the order contact synthesized.\*

For higher order derivatives we need only differentiate equation (3-9) further as given by equations (4-33, 44, 45), take the coefficients from Table (4-1) and proceed as outlined.

---

\*The corresponding  $E_{pq}$ 's are found in Table (4-1) where  $\beta = 0^\circ$  for the circle.

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} h_{12} \\ h_{13} \end{bmatrix}^{-1} \begin{bmatrix} -[E_{11}]_1 \\ -[E_{12}]_1 \end{bmatrix}$$

Design and Speed of  
Synthesis for Link Mechanisms

It often becomes desirable to synthesize the pin joints of the coupler link or fixed pivots of a planar mechanism pair while satisfying seven multiply separated precision points in coupler motion. The designer is often faced with such requirements when the solution to a position dimension size limitations or optimal constrained locations.

It is important here that one determines just what can be prescribed while satisfying seven precision points in coupler motion. For seven precision points there are six angles  $\gamma_i$  which indicate that there are six prescribable variables for the seven precision point study. For higher order studies, the angles would be replaced by the relative change of the linklengths

$$b_{a_1} \alpha^k + b_{a_2} \beta^k = 0 \quad (3-34)$$

as explained earlier in this chapter, and given by equation (3-7). The six prescribable variables may be any combination or variation of the following

$$\begin{aligned} & (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2), (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2) \\ & (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2), (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2) \\ & (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2), \left[ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right], \left[ \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right], (a, \beta) \\ & (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2), (b_{a_1}, \alpha_1), (b_{a_2}, \alpha_2) \end{aligned}$$

that can be gained by the versatility of specifying constraint requirements while satisfying seven multiple separated precision points.

The generalized axis constraint pair equation (3-4) is rewritten here as

$$a_{1j} \sin^3 \gamma_j + a_{1j} \sin^2 \gamma_j + a_{1j} \sin \gamma_j + a_{1j} \sin \gamma_j + a_{1j} = 0 \quad (3-17)$$

where

$$a_{1j} = f(a_1, \gamma, \theta, \theta_1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \theta, \theta_1, \theta_1$$

as defined by equations (3-5). Unlike the axis constraint pair synthesis, too study herein is concerned with synthesizing variables for both constraint pairs. Therefore, for each precision point  $j$  there exist two functions of the form

$$\text{constraint II } (f_{1j} \sin^3 \gamma_j + f_{1j} \sin^2 \gamma_j + f_{1j} \sin \gamma_j + f_{1j} \sin \gamma_j + f_{1j} = 0$$

$$\text{constraint II } (g_{1j} \sin^3 \gamma_j + g_{1j} \sin^2 \gamma_j + g_{1j} \sin \gamma_j + g_{1j} \sin \gamma_j + g_{1j} = 0 \quad (3-18)$$

where

$$f_{1j} = f(a_1, \gamma, \theta, \theta_1, \theta_1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta, \theta_1, \theta_1)$$

$$g_{1j} = f(a_1, \gamma, \theta, \theta_1, \theta_1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \theta, \theta_1, \theta_1)$$

as taken from equations (3-4) and (3-5).

The resolution of these two differential relations seems to be a formidable task. And since the algorithm requires the sets of  $\delta_{pq}$  and  $\eta_{pq}$  which have a common root or roots, a mode of solution becomes necessary, employing Sylvester's Resultant Method gives the resultant

$$R_1 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & 0 & 0 & 0 \\ 0 & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & 0 & 0 \\ 0 & 0 & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & 0 \\ 0 & 0 & 0 & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ \eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & 0 & 0 & 0 \\ 0 & \eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & 0 & 0 \\ 0 & 0 & \eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} & 0 \\ 0 & 0 & 0 & \eta_{1,1} & \eta_{1,2} & \eta_{1,3} & \eta_{1,4} & \eta_{1,5} \end{vmatrix} = 0, \quad (5-18)$$

This is to say that the determinant  $R_1$  must be singular for each of the precision points prescribed. Therefore one can prescribe any six of the following variables in addition to the coordinates of the seven precision points

$$\left\{ \begin{array}{l} \eta_{1,1}, \eta_{1,2}, \eta_{1,3}, \eta_{1,4}, \eta_{1,5}, \eta_{1,6} \\ \eta_{2,1}, \eta_{2,2}, \eta_{2,3}, \eta_{2,4}, \eta_{2,5}, \eta_{2,6} \end{array} \right\} \quad (5-19)$$

by employing a numerical search routine satisfying equations (5-18).

For the infinitesimal positions equations (5-6b) and the derivatives of the constraint equations (5-3) give

$$\text{Constraint (2)} \quad (x_{11}^k b_{11})a^k + (x_{12}^k b_{12})a^k + (x_{13}^k b_{13})a^k = 0$$

$$\text{Constraint (3)} \quad (x_{11}^k b_{11})a^k + (x_{12}^k b_{12})a^k + (x_{13}^k b_{13})a^k = 0$$

$$\text{Prescribed Order} \quad (b_{11}a^k + b_{12}a^k) = 0$$

(5-12)

where the coefficients  $b_{1ij}$  are taken from Table 4-1.

Therefore the requirements for any order of  $k$  become

$$a_1 = \begin{bmatrix} (x_{11}^k b_{11}) & (x_{12}^k b_{12}) & (x_{13}^k b_{13}) \\ (x_{11}^k b_{11}) & (x_{12}^k b_{12}) & (x_{13}^k b_{13}) \\ b_{11} & b_{12} & 0 \end{bmatrix} = 0. \quad (5-13)$$

Thus for any finite or infinitesimal precision point  $i$ , the requirements imposed by the prescribed precision point's coordinates and the design's prescribed variables equations (5-10) are given by equation (5-13) or (5-12) depending on the case study.

#### Coupler and Fixed Link Synthesis for Burmester Coupler

As was seen earlier from previous discussion, the synthesis of coupler and fixed links for the Burmester coupler represents a simplified form of the code constraint algorithm. Rewriting the generalized Burmester constraint equation, equation (3-17), for the finite case

$$a_{11} \cos \gamma_1 + a_{12} \cos \gamma_2 + a_{13} = 0 \quad (5-14)$$

where  $a_{ij}$  are defined by equations (5-28). Substituting equations for each constraint, results in the two equations

$$\begin{aligned} \text{Constraint 1: } (a_{11} \cos \gamma_1 + a_{12} \cos \gamma_2 + a_{13}) &= 0 \\ \text{Constraint 2: } (a_{21} \cos \gamma_1 + a_{22} \cos \gamma_2 + a_{23}) &= 0 \end{aligned} \quad (5-29)$$

where

$$\begin{aligned} a_{1j} &= f(a_1, v_1, R_1, R_2, \alpha, \beta) \\ a_{2j} &= f(a_2, v_2, R_1, R_2, \alpha, \beta) \\ &\quad \begin{matrix} j = 1, 2, 3, 4 \\ j = 1, 2, 3 \end{matrix} \end{aligned}$$

as taken from equations (5-28). Equations (5-29) require

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 + \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}^2 + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}^2 = 0 \quad (5-30)$$

or

$$(a_{11}a_{22} - a_{12}a_{21})^2 + (a_{11}a_{23} - a_{13}a_{21})^2 + (a_{12}a_{23} - a_{13}a_{22})^2 = 0 \quad (5-31)$$

By synthesizing the coupler link variables

$$\{(v_1, v_2, R_1, R_2)\}$$

will give the unknown variables of equation (5-28)

$$\begin{aligned} \text{and} \quad \left\{ \begin{array}{l} a_{1,1} \\ a_{1,2} \\ a_{1,3} \\ a_{1,4} \\ a_{1,5} \\ a_{1,6} \\ 1 \end{array} \right\} &= 0 \end{aligned} \quad (5-27)$$

where

$$a_{1,j} = \lim_{\lambda \rightarrow 0} \left\{ \begin{array}{l} a_{1,j}^{\lambda} \\ a_{1,j}^{\lambda} \\ a_{1,j}^{\lambda} \\ a_{1,j}^{\lambda} \\ a_{1,j}^{\lambda} \\ a_{1,j}^{\lambda} \\ 1 \end{array} \right\} = 0 \quad (5-28)$$

and

$$a_{1,j} = (a_{1,j}, u_1, u_1, v_1, u_2, v_2).$$

By specifying the fixed link variables

$$(a_{1,2}, a_{1,3}), (a_{1,4}, a_{1,5})$$

provided from equations (5-24)

$$\begin{aligned} \text{and} \quad \left\{ \begin{array}{l} a_{1,1}^{\lambda} \\ a_{1,2}^{\lambda} \\ a_{1,3}^{\lambda} \\ a_{1,4}^{\lambda} \\ a_{1,5}^{\lambda} \\ a_{1,6}^{\lambda} \\ 1 \end{array} \right\} &= 0 \end{aligned} \quad (5-29)$$

where

$$Q_{1j} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ 1 \end{bmatrix} \quad (5-20)$$

and

$$Q_{1j} = \{ \cos \alpha, \sin \alpha, \cos \beta, \sin \beta, \cos \gamma, \sin \gamma \}.$$

Therefore we need only satisfy the above requirements for synthesis of either the coupler link as fixed link, as for the infinitesimal displacements the procedure is identical to that outlined by equations (5-21) and (5-22).

## CHAPTER VI

### THE GENERALIZED DEWITT-FOSTER AND CARLITZ METHOD

In chapter II the generalized Forrester equation (2-7) was given as

$$\begin{aligned} & \lambda_{12} Q_{11} + \lambda_{13} Q_{12} + \lambda_{14} Q_{13} + \lambda_{15} (\omega Q_{11} + \nu Q_{12}) \\ & + \lambda_{21} (\omega Q_{11} + \nu Q_{12}) + \lambda_{22} Q_{21} + \lambda_{23} Q_{22} = E. \end{aligned} \quad (6-1)$$

Making the following substitutions

$$\begin{aligned} Q_{11} &= z_1 & (\omega Q_{11} + \nu Q_{12}) &= z_2 \\ Q_{12} &= z_3 & (\omega Q_{12} + \nu Q_{13}) &= z_4 \\ Q_{13} &= z_5 & Q_{21} &= z_6 \\ & & Q_{22} &= z_7 \end{aligned} \quad (6-2)$$

gives the linearized expressions

$$\sum_{n=1}^7 \lambda_{n1} z_n = E \quad (6-3)$$

in the required chamber coordinate formulation. The six post-positions of the working plane are  $k = 1, 2, 3, 4, 5$  where the first position,  $k = 1$ , is superposed with the fixed origin. The last five of these positions may be expressed as

$$\frac{1}{\sin^2} \left| \frac{1}{\sin^2} \right| = 0, \quad (4-3)$$

In prescribing six point-positions of the moving plane, as illustrated in Figure (4-1), the angular rotation of two of the positions must be compatible so that all combinations of five positions have two common Bertrand pairs. Similarly, infinitesimal cases require identical signposts compatibility.

#### Theorem I

There exists a unique pair of Bertrand constraints (pair) for six mutually separated position in cylinder motion.

#### Proof

For six mutually separated positions, equations (4-1) can be reconstructed in terms of 3 unknowns. This can be accomplished by Gauss-Jordan reduction or by Cramer's rule. This shows one to establish the two equations

$$q_1 \frac{1}{\sin^2} + q_2 \frac{1}{\sin^2} + q_3 \frac{1}{\sin^2} = 0 \quad (4-4)$$

and

$$h_1 \frac{1}{\sin^2} + h_2 \frac{1}{\sin^2} + h_3 \frac{1}{\sin^2} = 0$$

where

$$q_1, h_1 = \frac{1}{\sin^2}, \frac{1}{\sin^2}, \frac{1}{\sin^2}$$

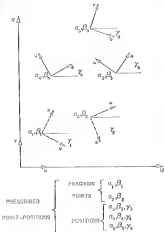


Figure (8-1) Five Relatively Separated Point-Positions in Coplanar Space

one direct bounding condition that all of the Berthelot points are, namely, the line defined by equation (8-1a) and all of the optimum points satisfy equation (8-1a). From Chapter 11 it was shown that five positions uniquely determine the Berthelot points as the intersection of two circles (the centers may also be determined by two circles). Since the line defined by equations (8-1) can only intersect a circle twice, there exists a unique pair of Berthelot coordinates satisfying six mutually separated positions in coplanar motion.

#### Algorithm for Six Mutually Separated Point Positions in Coplanar Motion

For simplicity the following algorithm will be presented in the most general sense, i.e. it is given in terms of the requirements for all cases. The formulation will show that each case study requires a different set of analytical expressions to define the compatibility region or the derivative's magnitude depending on the case being studied.

Rearranging equation (8-6) into the form

$$\begin{aligned}
 &A_1 \frac{x}{x_1} + A_2 \frac{x}{x_2} + A_3 \frac{y}{y_1} + A_4 \frac{y}{y_2} + A_5 \frac{z}{z_1} \\
 &\quad - A_6 \frac{x}{x_1} - A_7 \frac{y}{y_1} \quad i = 0, 1, 2, 3, 4, 5
 \end{aligned}
 \tag{8-4}$$

and applying Cramer's Rule given, after a sign adjustment,

$$\begin{aligned}
T_{11}^2 &= T_{11}^2 T_{11}^2 + T_{12}^2 T_{12}^2 \\
T_{12}^2 &= T_{21}^2 T_{21}^2 + T_{22}^2 T_{22}^2 \\
T_{13}^2 &= T_{31}^2 T_{31}^2 + T_{32}^2 T_{32}^2 \\
T_{14}^2 &= T_{41}^2 T_{41}^2 + T_{42}^2 T_{42}^2 \\
T_{15}^2 &= T_{51}^2 T_{51}^2 + T_{52}^2 T_{52}^2
\end{aligned}
\quad (6-7)$$

where

$$\begin{aligned}
T &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{11} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{12} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{13} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{14} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{15} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \\
T_{16} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}
\end{aligned}
\quad (6-8)$$

$$i = 0, 1, 2, 3, 4, 5$$

As shown by relations (1), the identities

$$T_1 T_2 = T_3 T_4 + T_5 T_6$$

and

$$T_3 T_4 = T_5 T_6 + T_7 T_8$$

must be satisfied. Substituting equations (6-7) into

(6-8) gives the two quadratics

$$\begin{aligned}
 a_{11} \bar{z}_1^2 &= a_{11} \bar{z}_1 \bar{z}_1 + a_{11} \bar{z}_1^2 = 0 \\
 a_{21} \bar{z}_1^2 &= a_{21} \bar{z}_1 \bar{z}_1 + a_{22} \bar{z}_1^2 = 0
 \end{aligned}
 \quad (8-10)$$

where

$$\begin{aligned}
 a_{11} &= \overline{P}_{11} + \overline{P}_{12} \overline{P}_{21} \\
 a_{12} &= (\overline{P}_{11} \overline{P}_{12} - \overline{P}_{22}) + (\overline{P}_{12} \overline{P}_{11} + \overline{P}_{11} \overline{P}_{22}) \\
 a_{13} &= \overline{P}_{12} \overline{P}_{12} - \overline{P}_{21}^2 \\
 a_{14} &= \overline{P}_{21}^2 - \overline{P}_{22} \overline{P}_{12} \\
 a_{22} &= (\overline{P}_{11} \overline{P}_{12} - \overline{P}_{22}) + (\overline{P}_{11} \overline{P}_{12} + \overline{P}_{22} \overline{P}_{12}) \\
 a_{23} &= -(\overline{P}_{12}^2 + \overline{P}_{12} \overline{P}_{12}).
 \end{aligned}
 \quad (8-11)$$

From equations (8-11), the identity

$$a_2 = \frac{a_1}{a_3}$$

shows that equations (8-10) can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{Bmatrix} \bar{z}_1^2 \\ \bar{z}_1 \\ 1 \end{Bmatrix} = 0.
 \quad (8-12)$$

Equations (8-12) must have the same roots as determined by Theorem I so that

$$[a_1]^k = 0 \quad (8-13)$$

or

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}^k = 0 \quad (8-14)$$

where

$$\begin{aligned} \alpha_{pq} &= f(\alpha_{m_1}) \\ \alpha &= 1, 1, 2, 3, 4, 5, 6 \\ \beta &= 1, 2, 3, 4, 5. \end{aligned}$$

This allows one to express the singularity constrained functions as

$$F_1(\alpha_{m_1}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0 \quad (6-15)$$

and

$$F_2(\alpha_{m_1}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

where

$$a_{ij} = f(a_1^k, a_2^k, \gamma_j)$$

depending on the case of study.

Once these singularity constrained functions are established the quadratic equations (5-12) become defined and the Hurwitz polynomials are related by

$$a_1 = \frac{-a_{11}}{2a_{11}} \pm \frac{\sqrt{a_{11}^2 - 4a_{11}a_{22}}}{2a_{11}} \quad (6-16)$$

and

$$a_2 = \frac{-a_{11}}{2a_{11}} \pm \frac{\sqrt{a_{11}^2 - 4a_{11}a_{22}}}{2a_{11}}$$

and the  $v$  coordinates are given (8)

$$v_{12} = \frac{v_{11} + v_{13} u_{12}}{v} \quad (8-17)$$

is taken from equation (8-7). The coefficients of the conics are related by

$$B_{12} = - \left[ \frac{v_{11} + v_{13} u_{12}}{v} \right]$$

and (8-18)

$$B_{22} = - \left[ \frac{v_{11} + v_{13} u_{12}}{v} \right],$$

Thus one has gained the versatility of specifying four coplanar positions and two precision points for the moving plane. This study will be referred to as a point-position study since it is a combination of coplanar positions and precision points.

I	PRESCRIBED			CONSTITUTE
	$\alpha$	$\beta$	$\gamma$	
0	0.0	0.0	0°	-
1	-0.074	-0.024		30°
2	-0.076	-0.003		45°
3	0.00	-0.006	70°	-
4	1.007	0.000	180°	-
5	0.413	0.000	180°	-

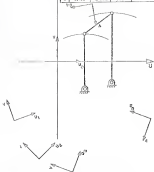


Figure 18-2) Mechanism Satisfying Six Multiply Separated Force-Positions in Coplanar Motion

## CHAPTER VII

### SINE MECHANISM SYNTHESIS PROCEEDS FINDER IN OPTIMAL MANNER

From Chapter II it was shown that five positions of the moving plane determine a mechanism of four-bar member pairs, and the four-bar member pairs are six combinations of two pairs. These six combinations provide six coupler curves corresponding to the five prescribed positions of the moving plane. If one employs the five position criteria for synthesizing sine producing points of the moving plane, the number of coupler curves (maximum of six) presents somewhat of a problem in optimal structural methods. This problem is the correspondence of the initial set of mechanisms (maximum of six) to the next revised set of mechanisms (also a maximum of six). As shown in Figure (7-1) for the first revised set 1A, 2A, 3A and 4A and the second revised set 1B and 2B, one can not distinguish if link 1B is the result of a transition from 1A or 2A or if 1B is due to a transition from 3A, 3A or 4A. If the wrong interpretation is made, the past history of the initial set of mechanisms becomes insignificant causing the convergence criteria to be variable. Other problems entail the measure of the error; i.e., it becomes necessary to analyze the mechanism's coupler

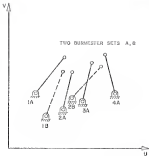


Figure (7.1) Deviation of a multivariate test for different inputs

given. The analysis of an eight-point position in terms of the error of the prescribed precision points is itself a very involved problem. Therefore, it is not desirable to employ the five position technique in synthesizing nine precision points of the moving plane.

In Chapter VI, theorem I shows that there is a unique coupler curve for six point-positions of the moving plane. It will be shown in what follows, that equations allow the six-point problem to look at the nine precision point problem in a purely analytical manner in terms of five functions and five unknowns. Although the following algorithm will be given in a general sense as made up of eight functions and eight unknowns, the problem can be reduced to five functions and five unknowns using trigonometric identification.

The six point-position problem was established by equations (7-10) as two quadratic functions

$$\begin{aligned} a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_2^2 + 0 \\ a_{21}x_1^2 + a_{22}x_1x_2 + a_{23}x_2^2 + 0 \end{aligned} \quad (7-11)$$

where

$$a_{pq} = f(\theta_{p1}, \theta_{p2})$$

are given by equations (6-11). Also from Chapter VI, the singularity constraint functions for the quadratics are given as

$$P_1(O_{20}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

and (2-1)

$$P_2(O_{20}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

of size

$$L_{ijk} = \{ (a_{ij}^k, b_{ij}^k, c_{ij}^k) \} \quad i = 1, 2, 3, 4, 5$$

depending on the case study. For the six point-position synthesis, the linear set is given by equations (5-3)

as

$$\begin{aligned} EL_1 &= E_{11}Z_1 + E_{12}Z_2 \\ EL_2 &= E_{21}Z_1 + E_{22}Z_2 \\ EL_3 &= E_{31}Z_1 + E_{32}Z_2 \\ EL_4 &= E_{41}Z_1 + E_{42}Z_2 \\ EL_5 &= E_{51}Z_1 + E_{52}Z_2 \end{aligned} \quad (7-10)$$

where

$$E_{ij} = f(a_{ij}^k) \quad i = 1, 2, 3, 4, 5$$

For nine precision points there must also be four separable linear sets for  $E_{1,2,3,4,5}$  in terms of  $E_{1,2}$ . The first relation of equations (7-10) defines the line passing thru the Freudenstein points. Therefore, for the four linear sets the first relation of each set must all

are coincident or identical. If, considering  $\mathbf{E}$  as the last prescribed position of each of the four linear axes, all having positions 0,1,2,3,4 in common, gives

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_0 + \mathbf{E}_2 \\ \mathbf{E}_2 &= \mathbf{E}_0 + \mathbf{E}_3 \\ \mathbf{E}_3 &= \mathbf{E}_0 + \mathbf{E}_4 \\ \mathbf{E}_4 &= \mathbf{E}_0 + \mathbf{E}_5 \end{aligned} \quad (7-6)$$

From these four relations there are six orthogonality conditions expressed by

$$\left. \begin{aligned} \mathbf{E}_1(\mathbf{E}_0) &= \begin{bmatrix} \mathbf{E} & \mathbf{E}_{12} \\ \mathbf{E} & \mathbf{E}_{13} \end{bmatrix} = 0 \\ \mathbf{E}_2(\mathbf{E}_0) &= \begin{bmatrix} \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{23} & \mathbf{E}_{24} \end{bmatrix} = 0 \end{aligned} \right\} \quad k=1,2,3,4,5,6$$

$$\left. \begin{aligned} \mathbf{E}_1(\mathbf{E}_2) &= \begin{bmatrix} \mathbf{E} & \mathbf{E}_{12} \\ \mathbf{E} & \mathbf{E}_{13} \end{bmatrix} = 0 \\ \mathbf{E}_2(\mathbf{E}_3) &= \begin{bmatrix} \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{23} & \mathbf{E}_{24} \end{bmatrix} = 0 \end{aligned} \right\} \quad k=1,2,3,4,5,6$$

$$\left. \begin{aligned} \mathbf{E}_1(\mathbf{E}_3) &= \begin{bmatrix} \mathbf{E} & \mathbf{E}_{12} \\ \mathbf{E} & \mathbf{E}_{13} \end{bmatrix} = 0 \\ \mathbf{E}_2(\mathbf{E}_4) &= \begin{bmatrix} \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{23} & \mathbf{E}_{24} \end{bmatrix} = 0 \end{aligned} \right\} \quad k=1,2,3,4,5,6.$$

$$\left. \begin{aligned} \mathbf{E}_3(\mathbf{E}_4) &= \begin{bmatrix} \mathbf{E}_{12} & \mathbf{E}_{13} \\ \mathbf{E}_{23} & \mathbf{E}_{24} \end{bmatrix} = 0 \end{aligned} \right\} \quad (7-7)$$

Thus all  $f_{ij}(x)$  in addition to the ten expressed by equations (7-11) give a total of eight functions. For nine finite positions there are eight angles of the moving plane. Therefore, the number of variables (angles) equals the number of functions thus providing a closed set of conditions.\*

If positions 1, 7 and 8 are finite positions of the moving plane, the singularity functions are related by

$$\begin{aligned} X_1 &= b_{11} \sin \theta_1 + b_{12} \cos \theta_1 + b_{13} \theta_1 \\ X_2 &= a_{11} \sin \theta_1 + a_{12} \cos \theta_1 + a_{13} \theta_1 \\ X_3 &= b_{11} \sin \theta_1 + b_{12} \cos \theta_1 + b_{13} \theta_1 \\ X_4 &= a_{11} \sin \theta_1 + a_{12} \cos \theta_1 + a_{13} \theta_1 \\ X_5 &= b_{11} \sin \theta_1 + b_{12} \cos \theta_1 + b_{13} \theta_1 \\ X_6 &= a_{11} \sin \theta_1 + a_{12} \cos \theta_1 + a_{13} \theta_1 \end{aligned} \quad (7-12)$$

where

$$b_i, a_i|_{pq} = f(x_{pq}) \quad i = 1, 2, 3, 4, 5$$

and may be expressed by

$$\begin{aligned} b_{11} &= -a_2 d_{11} + d_{11} \\ b_{12} &= b_2 d_{11} + d_{11} \\ b_{13} &= 1/2(a_2^2 + d_{11}^2) d_{11} + a_2 d_{11} \\ &\quad (b_2 d_{11} + d_{11}) \end{aligned}$$

\*To the same extent as for the finite angles, the absolute values of the derivatives become variables.



$$\begin{aligned}
 r_1(A_{12}) &= -(\alpha_{12}r_{12} - \alpha_{13}r_{13})^2 + (\alpha_{22}r_{22} - \alpha_{23}r_{23})^2 \\
 &\quad + (\alpha_{32}r_{32} - \alpha_{33}r_{33})^2 = 0 \\
 k &= 1, 2, 3, 4, 5
 \end{aligned}
 \tag{7-9}$$

or satisfying the identity

$$\sin^k \gamma_1 + \cos^k \gamma_2 = 1. \tag{7-10}$$

Combining the singularity functions (7-7) with (7-9) gives

$$\begin{aligned}
 r_1(A_{12}) &= 0 \\
 r_2(A_{12}) &= 0 \\
 r_3(A_{12}) &= 0 \\
 r_4(A_{12}) &= 0 \\
 r_5(A_{12}) &= 0 \\
 k &= 1, 2, 3, 4, 5.
 \end{aligned}
 \tag{7-11}$$

Therefore, the nine precision point synthesis problem has been reduced into a five parameter problem and allows the kinematician to look at the problem as a purely analytical function. Figure (7-2) illustrates a mechanism satisfying nine prescribed positions points for the finite case.

By employing the kinematic identity (7-10), the method is restricted to a set of nine precision points having a maximum of three finite angles or four finite positions. The method may, however, be extended to include all orders of contacts by linearly restricting the absolute values of the derivatives. It is felt that these

requirements are not warranted in the presentation of the evidence and shall not be discussed in this text. The following cases are satisfied by equations (7-11).

CASE		COMBINATION			
1	FFFFF	F	F	F	
2	FFFFF	FF	F	F	
3	FFFFF	F	F	F	F=F
4	FFFF	FFF	F	F	F
5	FFFF	FF	FF	F	F
6	FFFF	FF	F	F	F=F
7	FFFF	F	F	F	F=F=F
8	FFF	FFF	FF	F	F
9	FFF	FF	FF	FF	FF
10	FFF	FF	FF	FF	F=F
11	FFF	FF	F	F	F=F=F
12	FFF	F	F	F	F=F=F=F
13	FF	FF	FF	FF	FF=F
14	FF	FF	FF	FF	F=F=F
15	FF	FF	F	F	F=F=F=F
16	FF	F	F	F	F=F=F=F
17	F	F	F	F	F=F=F=F=F

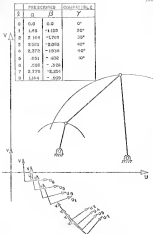


Figure 7-2 Mechanism Satisfying Three Multiply Separated Precision Points in Coupler Motion.

### Synthesis of Nine Type-II and Precision Points

Previous research by Polansky, Freudenstein and Gordon [14], Tena and Sparks [15] and Sparks [16] investigated the cases of five and six symmetrical displacements of the moving plane. As shown by these researchers, when the moving plane's positions have a characteristic symmetry about some axis, the corresponding coupler pairs are located in a characteristic pattern about the axis of symmetry. Since a complete description of symmetry can be found in references [15] and [16] as representation of the reductions of the codes, equations (2-18), will be presented in this text. This text will, however, extend the requirements for symmetry to nine precision points (see due to Symmetry).

One may employ the five position algorithm found in Chapter II in synthesizing nine symmetrical precision points of the moving plane. As will be shown in what follows, this is a result of conditions imposed on four of the positions located on one side of Y axis. For simplifying it is advantageous to separate the symmetrical mechanism into three distinct cases and their requirements.

#### Case 1

The four desired points and their centers are located on the Y axis. These requirements are given by

$$C_p = 0$$

$$E_{ij} = 0 \quad i = 2, 3.$$

It can be shown that these conditions are satisfied when

$$\begin{aligned} G_{11} &= \begin{vmatrix} A_{111} & A_{112} & A_{113} & A_{114} \end{vmatrix} = 0 \\ G_{12} &= \begin{vmatrix} A_{211} & A_{212} & A_{213} & A_{214} \end{vmatrix} = 0 \\ G_{13} &= \begin{vmatrix} A_{311} & A_{312} & A_{313} & A_{314} \end{vmatrix} = 0 \\ G_{14} &= \begin{vmatrix} A_{411} & A_{412} & A_{413} & A_{414} \end{vmatrix} = 0 \\ & i = 1, 2, 3, 4. \end{aligned}$$

### Case II

The Eulerian points and their centers are symmetrically located about the  $V$  axis. These requirements are given by

$$\begin{aligned} u_1 &= -u_2 \\ v_1 &= v_2 \\ E_1 &= -E_2 \\ E_3 &= E_4. \end{aligned}$$

It can be shown that these conditions are satisfied when

$$\begin{aligned} G_{11} &= \begin{vmatrix} A_{111} & A_{112} & A_{113} & A_{114} \end{vmatrix} = 0 \\ G_{12} &= \begin{vmatrix} A_{211} & A_{212} & A_{213} & A_{214} \end{vmatrix} = 0 \\ G_{13} &= \begin{vmatrix} A_{311} & A_{312} & A_{313} & A_{314} \end{vmatrix} = 0 \\ G_{14} &= \begin{vmatrix} A_{411} & A_{412} & A_{413} & A_{414} \end{vmatrix} + \begin{vmatrix} A_{111} & A_{112} & A_{113} & A_{114} \\ A_{211} & A_{212} & A_{213} & A_{214} \\ A_{311} & A_{312} & A_{313} & A_{314} \\ A_{411} & A_{412} & A_{413} & A_{414} \end{vmatrix} + \\ & \begin{vmatrix} A_{111} & A_{112} & A_{113} & A_{114} \\ A_{211} & A_{212} & A_{213} & A_{214} \\ A_{311} & A_{312} & A_{313} & A_{314} \\ A_{411} & A_{412} & A_{413} & A_{414} \end{vmatrix} + \begin{vmatrix} A_{111} & A_{112} & A_{113} & A_{114} \\ A_{211} & A_{212} & A_{213} & A_{214} \\ A_{311} & A_{312} & A_{313} & A_{314} \\ A_{411} & A_{412} & A_{413} & A_{414} \end{vmatrix} = 0 \\ & i = 1, 2, 3, 4, \end{aligned}$$

### Case 11)

The parameter points are located on the  $x$  axis and their centers on the  $y$  axis. These requirements are given by

$$\begin{aligned}v_n &= 0 \\ \theta_n &= 0 \\ n &= 1, 2.\end{aligned}$$

It can be shown that these conditions are satisfied when

$$\begin{aligned}C_{n1} &= \begin{vmatrix} A_{n1} & A_{n2} & A_{n3} & A_{n4} \end{vmatrix} = 0 \\ C_{n2} &= \begin{vmatrix} A_{n1} & A_{n2} & A_{n3} & A_{n4} \end{vmatrix} = 0 \\ C_{n3} &= \begin{vmatrix} A_{n1} & A_{n2} & A_{n3} & A_{n4} \end{vmatrix} = 0 \\ C_{n4} &= \begin{vmatrix} A_{n1} & A_{n2} & A_{n3} & A_{n4} \end{vmatrix} = 0 \\ & \quad j = 1, 2, 3, 4.\end{aligned}$$

In the synthesis of nine symmetrical precision points of the moving plane it is necessary to satisfy the conditions of one of the three axes for the angles  $\gamma_k$  or the derivative's magnitude. Satisfying these conditions will give the same displacements in the opposite coordinates of the fixed plane as illustrated in Figures 13-15.



## APPENDIX

# APPENDIX A

## EXPANSION OF THE CIRCUITORY EQUATION BY ITERATION

The problem of expanding the compound square matrix for four multiply separated positions as separate series is best represented by a method of determinants.

The compound square matrix for four positions is given by

$$\begin{bmatrix} (A_{11} + A_{12}u + A_{13}v)(A_{11} + A_{12}u - A_{13}v)(A_{11} + A_{12}u + A_{13}v) \\ (A_{11+1} + \quad \quad + 1)(A_{11+1} + \quad \quad + 1)(A_{11+1} + \quad \quad + \quad \quad) \\ (A_{11+1} + \quad \quad + 1)(A_{11+1} + \quad \quad + 1)(A_{11+1} + \quad \quad + \quad \quad) \end{bmatrix} \begin{Bmatrix} (u_1) \\ (u_1) \\ (u_1) \end{Bmatrix} = 0 \quad (A-1)$$

where the coefficients  $A_{ij}$  are found in table A-1. A resulting general compound cubic may be written as

$$A_1 u^3 + (P_1 + P_2 v)u^2 + (Q_1 + R_1 v + R_2 v^2)u + (Q_1 + P_2 v + R_2 v^2 + R_3 v^3) = 0. \quad (A-2)$$

The coefficients  $R$  are obtained by expanding the compound square matrix to give

$$R_1 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \end{vmatrix} +$$

$$\begin{aligned}
D_1 &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix} \\
D_2 &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{11} & A_{12} & A_{13} \end{vmatrix} + \\
D_3 &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{vmatrix} + \\
&\quad \begin{vmatrix} A_{11} & A_{12} & -A_{13} \\ A_{21} & A_{22} & -A_{23} \\ A_{31} & A_{32} & -A_{33} \end{vmatrix} + \begin{vmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{vmatrix} + \\
&\quad \begin{vmatrix} A_{11} & -A_{13} & A_{12} \\ A_{21} & -A_{23} & A_{22} \\ A_{31} & -A_{33} & A_{32} \end{vmatrix} + \begin{vmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{vmatrix} + \\
D_4 &= \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix} + \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix} + \\
&\quad \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
D_5 &= \begin{vmatrix} A_{12} & A_{13} & A_{11} \\ A_{22} & A_{23} & A_{21} \\ A_{32} & A_{33} & A_{31} \end{vmatrix} \\
D_6 &= \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix} + \begin{vmatrix} A_{11} & -A_{13} & A_{12} \\ A_{21} & -A_{23} & A_{22} \\ A_{31} & -A_{33} & A_{32} \end{vmatrix} + \\
&\quad \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
D_7 &= \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix} + \begin{vmatrix} A_{11} & -A_{13} & A_{12} \\ A_{21} & -A_{23} & A_{22} \\ A_{31} & -A_{33} & A_{32} \end{vmatrix} + \\
&\quad \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
D_{11} &= \begin{vmatrix} A_{12} & -A_{13} & A_{11} \\ A_{22} & -A_{23} & A_{21} \\ A_{32} & -A_{33} & A_{31} \end{vmatrix}
\end{aligned} \tag{6-31}$$

where six of the determinants are singular since they have equal columns. From equations (6-31) it is also seen that

$$D_1 = D_2 + D_3 = D_{11}$$

is then the basic (characteristic equation) becomes

$$\begin{aligned} & (a_1 u + a_2 v)(u^2 + v^2) + a_3 u^2 + a_4 v^2 \\ & + a_5 uv + a_6 u + a_7 v + a_8 = 0 \end{aligned} \quad (8-4)$$

where

$$\begin{aligned} a_1 &= B_1 \\ a_2 &= B_{1,2} \\ a_3 &= B_3 \\ a_4 &= B_4 \\ a_5 &= B_5 \\ a_6 &= B_6 \\ a_7 &= B_7 \\ a_8 &= B_8 \end{aligned} \quad (8-5)$$

For the cubic equation (8-4) the circular property  $(B_1 u + B_2 v)(u^2 + v^2)$  determines the number of real asymptotes (see real asymptotes for the general linear position angles for the function).

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MEMORANDUM FOR THE RECORD

Mr. Wesley Sparks was born March 17, 1941, to William Henry and Mary Lynn Sparks in Okemah, Oklahoma. At a very early age his family moved to Birmingham, Alabama, where he completed his elementary school education. Upon graduating from Poncaville High School in 1961, he entered Oklahoma Junior College and received his Associate of Arts degree in 1963. He then transferred to the University of Florida and completed his Bachelor of Mechanical Engineering degree in August, 1965. In September, 1965, he entered graduate school under a research assistantship and completed his Master of Science in Engineering in August, 1967. He has continued his education in pursuit of a Doctor of Philosophy with major in Mechanical Engineering.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

  
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Professor of Mechanical Engineering

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This dissertation was submitted to the Dean of the College of Engineering and to the Graduate Council, and was accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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